4-10-2024

The Mathematical and Historical Significance of the Four Color Theorem

Brock Bivens

The Mathematical and Historical Significance of the Four Color Theorem

BROCK BIVENS
OUACHITA BAPTIST UNIVERSITY

SENIOR THESIS APPROVAL

This Honors thesis entitled

"The Mathematical and Historical Significance of the Four Color Theorem"

written by

Brock Bivens

and submitted in partial fulfillment of
the requirements for completion of
the Carl Goodson Honors Program
meets the criteria for acceptance
and has been approved by the undersigned readers.

____

Dr. Jeff Sykes, Thesis Director

____

Dr. Debra Coventry, Second Reader

____

Mr. Adam Wheat, Third Reader

____

Dr. Barbara Pemberton, Honors Program director

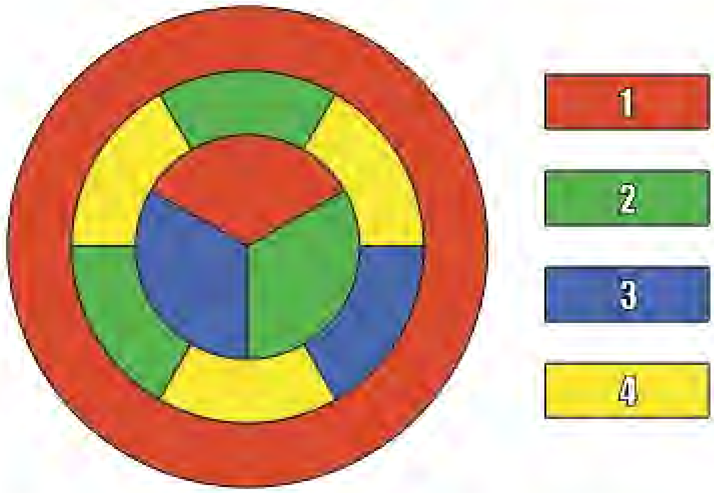
April 16, 2023

ABSTRACT

The Four Color Theorem as a whole has a unique and interesting background, but the recent proof of this theorem sparked global debate among mathematicians as to its legitimacy. A computer was used to prove this theorem, yet I insist that it was a legitimate proof. I interviewed multiple practicing mathematicians throughout my process researching this controversial theorem and expounded on their remarks throughout my thesis. They provided immense insight as to the culture surrounding mathematical proofs, the controversy of the four color theorem, and also their deep connections to the mathematicians who proved it.

INTRODUCTION

The Four Color Theorem basically says that you can color any map in a plane (2-dimensional plane) such that regions of the plane that share a common line (also referred to as an edge, other than a single point) don't share the same color. For illustration purposes, I have included an image below.



(Figure 1.1)

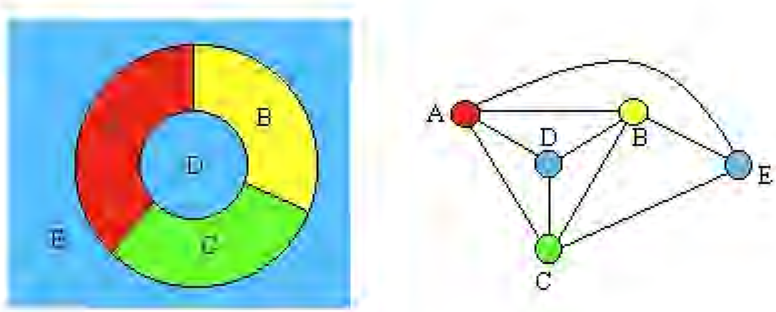
As you can clearly see, this two dimensional plane contains various polygons and they are all colored in such a way that it only took them four colors to color this plane. This, in essence, is the entire problem at hand. How on earth can we prove that it only takes 4 colors (or less) to color a two dimensional plane.

BACKGROUND

We begin our journey of the Four Color Theorem all the way back in 1852, where Francis Guthrie originally stated the conjecture for the Four Color Theorem that was titled “Guthrie’s Problem” at the time. Then, after that work that Guthrie conjectured, another mathematician by the name of Arthur Cayley wrote the first paper on this topic, where he expressed the major difficulties that proving this problem could have. Then, just a short year later another mathematician by the name of Alfred Bray Kempe published a solution that was included in the 1879 issue of the *American Journal of Mathematics* and this solution to Guthrie’s original problem was accepted for a little bit, until in 1890, yet another mathematician by the name of Pearcy John Heawood discovered a fatal flaw in Kempe's proof. This would ultimately leave the problem unsolved for around another 86 years. **(1)**

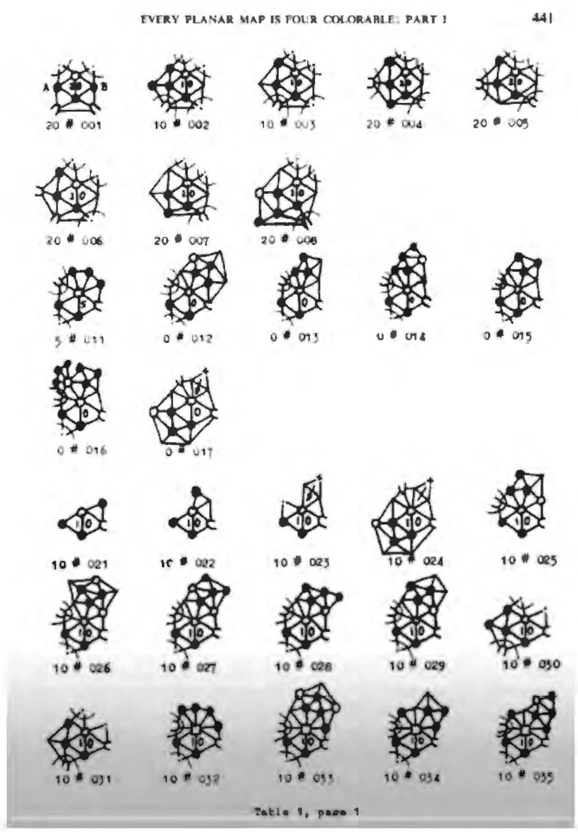
PROOF OF THE FOUR COLOR THEOREM

In 1977, two mathematicians from the University of Illinois named Kenneth Appel and Wolfgang Haken were eventually able to come up with a proof of the Four Color Theorem, but how did they do it? Their process in proving the Four Color Theorem came from an area of mathematics named Graph Theory. This is an area of math that is very helpful in illustrating complex ideas in a visual sense. Appel and Haken would use the techniques from graph theory to create all of these different possibilities of the Four Color Theorem, and all of the different cases that could ever possibly arise. Their process would go something like this. Take the same picture we saw earlier, and replace each part of the graph with a little circle. And then, whatever shapes are touching a shape that is next to them (a common vertex/line), draw a line between your little circles that would signify that they are "next" to each other. I will include an image from Math Pages below to further communicate this idea.



(Figure 2.1)

This is precisely what Appel and Haken would do, except they would use mathematical techniques to describe every possible combination of these little circles and lines and see if it was possible to prove the Four Color Theorem. I would also like to include some of their drawings below as well.



(Figure 2.2)

From a mathematical standpoint, they decided to represent these drawings using a branch of math called graph theory. Graph Theory proved very helpful in the actual proof of the Four Color Theorem because it allowed them to talk about these very complex ideas in terms of vertices and edges. A vertex , simply stated, is a commonly shared point of two shapes while an edge is a “line” that connects vertices. These two ideas would be critical in analyzing all of the different combinations of vertices and edges. I also believe that it is important to point out that the Four Color Theorem is defined for planar graphs, or more simply a two-dimensional plane, this is important because in higher dimensions there are cases where the Four Color Theorem isn't applicable.

I believe that this connection of these vertices and edges are extremely interesting due to the many different combinations of these graphs. There were over 1900 cases that Appel and Haken would have to discover, check, and make sure that no more existed. You may be thinking to yourself, how on earth could this have been solved in such a reasonable time? Well, something that hadn't been utilized in mathematics was the use of computers, because traditional mathematicians wanted to have all of their thinking and reasoning on paper, and that could be checked and critiqued by other mathematicians. Appel and Haken utilized the Teletype Model ASR-33 which only ran at a staggering 110 bits per second, which if you put that into perspective, would take that computer 105 days to download a 1 gigabyte file. So, because it would have taken Appel and Haken exponentially longer to compute all of these cases by hand, they utilized this computer in order to prove the Four Color Theorem. With the help of John Cocke, an IBM Researcher, who was on the team that developed the compiling system for their computer, they were able to more efficiently check the cases as well as publish a proof of the Four Color Theorem for the first time in history.

CONTROVERSY OF THE FOUR COLOR THEOREM PROOF

This would be where the chaos would erupt. Many older mathematicians at that time didn't really want to have computers in mathematics and believed that they should be separate, while some younger mathematicians were all for computer aided proofs. If a human couldn't check the validity of a proof in the first place, is the proof ever even true? One article I read even stated that the New York Times straight up refused to publish any news about this proof just because of the fact that "it was going to be wrong anyway." How could you trust a program that does all of these computations but can't explain the reasonings behind them? That is a core principle of mathematics, starting with basic assumptions and then using reasoning to determine more truths about specific objects or systems.

Personally, I think that given the scenario that Appel and Haken were in, they faced a lot of adversity given the proof and the fact that they used computers to help them solve this theorem, but they were the spark that helped further the area of mathematics as a whole. This development has proven so critical that even today, most of the mathematical work is done on a computer in LaTeX with more computers being used in proofs than ever before.

INTERVIEW WITH DR. SIBLEY

Throughout my time researching the Four Color Theorem , my thesis Advisor, Dr. Jeffery Sykes from Ouachita Baptist University was able to connect me with some great mathematicians. Namely, Dr. Thomas Sibley, Professor Emeritus of Mathematics from St. John's University in Minnesota. Dr. Sibley taught for well over 40 years at the collegiate level, but prior to that he taught high school mathematics in the Congo. Dr. Sibley has written many mathematical textbooks in various fields of math, namely in algebra, geometry, and logic. Dr. Sibley described himself as a mathematical logic researcher which was one of the main reasons I chose to interview him. He can help me gain insight into the logic behind what proofs are and how they are utilized in mathematics as a whole. When asked what a mathematical proof is, Dr. Sibley said that a mathematical proof is a “clear, complete, convincing argument based on explicit assumptions.” I think that this definition is so simple, yet contains so much beauty of mathematics as a whole. There are countless proofs that just seem so elegant yet tell such a beautiful picture, namely, Euler’s formula which relates pi, euler’s number, and the imaginary number *i*. Again, there are countless proofs and or formulas that have been developed that deserve countless papers over them, that Dr. Sibley would’ve loved to have discussed. I then began to discuss with Dr. Sibley about the published proof of the Four Color Theorem that was released in 1976, and he began to describe what all he was doing at the time, and what he thought about the proof. Dr. Sibley said “I was over in the Congo, and one of my friends told him about these people that had come up with a counterexample of the four color theorem, but it ended up being an April fools joke in 1974.” Then, after asking about the specific Four Color Theorem proof, Dr. Sibley said “ It was a big deal in 1976, I was in graduate school and my advisor in logic told me that computers in mathematics were a terrible idea. How can we stoop so low as to trust a computer with our gorgeous mathematical proofs?” I think that this was the overall belief among older mathematicians at the time, as the idea of using technology in mathematics was foreign. Mathematicians are known for being stubborn in their ways and ideas, but at the time they didn’t want to have to change the very way they studied, wrote, and learned mathematics. As Dr. Sibley’s advisor said, “How can we stoop so low as to trust a computer with our gorgeous mathematical proofs?” This was the overall consensus of most mathematicians at the time. Dr. Sibley went on to say that the first time he ever touched a computer was in 1982, 2 years after completing his PhD, where he typed his dissertation on a manual typewriter. To think of how far we have come as a society technology-wise, I find this to be fascinating. Dr. Sibley further said by the 1980's, many mathematicians were beginning to realize that the skeleton of the proof was solid and that in principle it should work (talking about the Four Color Theorem). During this time, other proofs were being published with the use of computers and that there was a shift that was happening among mathematicians to where computers were beginning to be seen as friends, and not enemies. In addition, Dr. Sibley mentioned these programs that were being published that could be used to check proofs in mathematics that were being done with computers. Like an HTML validator, but for mathematical proofs. These proof validators did require specific syntax, but the more they were developed, the more widespread they became. In addition, Dr. Sibley noted that when Appel and Haken published the proof, he felt as if it could've been right or it could've been wrong. He mentioned that it was more a question of probability. There were so many cases to check, and probability wise, it was probably correct, but Dr. Sibley wasn't really convinced at the time. His hope was that somebody could come up with a proof that somebody could "write and read and somebody else could check." Thankfully, with the more rapid expansion of technology, other computer scientists and mathematicians were able to begin to trust computer-aided proofs due to the logic behind writing the code for each proof. Overall, my time with Dr. Sibley was very insightful and validated some of the thoughts I had about computerized proofs back in the day. **(4)**

INTERVIEW WITH DR. GRAVER

In addition to interviewing Dr. Sibley from St. John's University, I had the great privilege to interview Dr. Jack Graver from Syracuse University. Dr. Graver grew up with the thought that he would never get into college due to an intellectual disability that he had, but hadn't been diagnosed at the time, and thus he was just viewed as "stupid" or "lazy." He ended up attending Miami University in Ohio, where he would earn his Bachelors in Mathematics. He then went on to the University of Indiana where he got his Masters and Doctoral Degree. From there, he taught at Dartmouth for a few years, and then at Syracuse where he would spend the rest of his career researching rigidity theory and the structure of large plane graphs. With over 35 publications, 15 articles, various NSF summer institutes, consulting for a university overseas, and even publishing his own book on rigidity theory. Similar to how I started my interview with Dr. Graver, I began our interview with the question about defining a proof in his terms. Dr. Graver's response varied in the sense that he defined a mathematical proof as a "logical argument that is accepted by the mathematical community." This was very interesting because Dr. Graver specifically mentioned the fact that it must be accepted by the mathematical community, which could've been biased based on our previous conversations leading up to the interview, but I do not wish to draw any hard conclusions. The question then arises, was the Four Color Theorem even a proof after it was published? I leave this question open for debate, but from there, I then continued to ask Dr. Graver about the Four Color Theorem. Dr. Graver mentioned how he was "at a conference at the University of Waterloo, where Dr. William Tutte, one of the most prolific graph theory researchers in the history of mathematics was leading this conference, and the minute that the announcement hit, the television stations from Toronto came all the way over to Waterloo to interview Dr. Tutte. The television reporters then put Dr. Tutte on the spot in front of all of these other mathematicians at the same conference, where Dr. Tutte would say that he accepted their proof after seeing it." Just hearing this story is inspiring for a young mathematician, how all of these people were so invested in the response of somebody they looked up to. I think that this is a sign of how everybody is always striving for knowledge and understanding within a complex subject. Dr. Graver's reaction was that because "Appel was a fabulous mathematician, and that Haken was a well known computer scientist that I accept the proof." If I were to have been in Dr. Graver's shoes I would've felt hesitant to trust somebody else's word without reading the proof for myself and collaborating with other mathematicians.If somebody with a ton of mathematical knowledge says something is true, then odds are it's true, but you can't be certain until you dive into the math yourself. He even went on to say that mathematicians at the conference began arguing about the validity of the proof and how Dr. Tutte was incorrect in saying that it was a valid proof. In the same manner, others on Dr. Tutte's side believed the proof to be valid. It's almost like there was a giant boxing match over this proof, but thankfully it wasn't as severe as Hippasus' case. Furthermore, Dr. Graver also explained how divided the mathematical community was on the utilization of computers in proof work, even after the conference. Dr. Graver's personal belief on the use of them at the time was that he was actually working with Ramsey Numbers and used a similar approach to Appel and Haken, in that they used Basic, a mathematical programming language in order to compute larger cases of these Ramsey numbers, and the computer ran all night and found solutions to all of these cases. After Dr. Graver used this language to code up a solution, he then accepted the proof of the Four Color Theorem by Appel and Haken. I think that given the specific scenario that Dr. Graver was in, he was more likely to accept the proof because he had seen firsthand how computer were used **(5)**

CONCLUSION

Interviewing both of these mathematicians was such an informative and excellent experience. I think that throughout the history of the Four Color Theorem, its proof has had great implications for mathematics as a whole. While Dr. Sibley wasn't necessarily sure as to the validity of the proof, Dr. Graver trusted the opinion of Dr. Tutte. I believe that at that moment in time that my opinion would've been much more skeptical, alongside Dr Sibley's point of view. No other proof had ever been written by a computer, and just due to the enormous amount of computation that these computers were having to do, I would've needed much more evidence and reasoning behind the program as a whole. In addition, I can also see the point of view that if one of the world's most brilliant graph theorists said that the proof was valid, I definitely couldn't stand up to him and say that he would be wrong. In addition, the use of computers in mathematical proofs skyrocketed, because now there were areas of math that could be developed and further studied due to the amount of computation that was now possible. The Four Color Theorem was eventually reproved by reducing the number of uncountable sets down to under 640, which was originally over 1900, while also improving the code that was used to check each case. All of these meant more efficient programs and a much more elegant and simpler proof. This theorem has helped develop some of the finest mathematics in the last half century mainly due to the wide acceptance of computerized proofs, even though originally, there was some controversy. Mathematicians began chatting with one another, reading articles published in journals, using computers for themselves, and even using proof validators that were developed by other working mathematicians at the time. All in all, the Four Color Theorem has been quite the groundbreaker for using computers in proofs, starting some serious mathematical tension, and also a historical problem that continues to amaze those who study it.

REFERENCES

(1) “Four-Color Theorem.” *From Wolfram MathWorld*, mathworld.wolfram.com/Four-ColorTheorem.html#:~:text=The%20four%2Dcolor%20theorem%20states,conjectured%20the%20theorem%20in%201852. Accessed 10 Dec. 2023.

(2) *The Four Color Problem and Its Connection to South African Flora*, ion.uwinnipeg.ca/~ooellerm/guthrie/FourColor.html. Accessed 10 Dec. 2023.

(3) “Four Color Theorem.” *Wikipedia*, Wikimedia Foundation, 13 Nov. 2023, en.wikipedia.org/wiki/Four_color_theorem.

(4) Sibley, Thomas. Interview. Conducted by Brock Bivens. 27 February 2023.

(5) Graver, Jack. Interview. Conducted by Brock Bivens. 2 March 2023.