5-1968

Plane Protective Geometry

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PLANE PROJECTIVE GEOMETRY

by

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Special Studies

May 16, 1968

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PLANE PROJECTIVE GEOMETRY

The following study was based on the text A Modern Introduction to Geometries by Annita Tuller, Associate Professor of Mathematics Hunter College. The study consisted of problem solving at the end of each topic studied. Therefore, this paper contains a brief summary of the topics covered followed by the problems solved with their respective drawings. No attempt is made to include all of the theorems, axioms, or definitions necessary to solve the problems but page references are given to refer to the text.

Geometric properties are classified into two categories: metric properties, which are those concerned with measurements of distances, angles, and areas, and descriptive properties, those concerned with the positional relations of geometric figures to one another. Descriptive properties are preserved in plane figures when a figure is projected from one plane onto another while metric properties may not be preserved.

Projective geometry was developed as an extension of Euclidean geometry, keeping the parallel postulate and adding a line to contain the ideal points. Two parallel lines had a unique ideal point and intersecting lines had two different ideal points. Later, projective geometry was seen to be independent of the theory of parallels and developed as an abstract science with its own set of axioms.

The first topic considered involved the axioms of incidence and the principle of duality. When a point and a line are incident the point lies on the line. Points incident with the same line are collinear, or lines incident with the same point are concurrent.

The next topic was the Desarguesian projective plane and discussed axioms and theorems concerned with Desargueses work. Harmonic sets are then discussed giving the definitions of a quadrangle, diagonal points, diagonal lines and the diagonal triangle of a quadrangle. Perspectivities and projectivities comprise the study for the problems following harmonic sets and last is the net of rationality.

PROBLEMS

#1. pg. 36. Exercise 1.
Show that the four axioms of incidence are consistent.

Refer to drawing 1. Axiom 1: There exists a point P and a line a that are not incident. Axiom 2: Every line a is incident with at least three points A, B, C. Axiom 3: Any two distinct points A, B are incident with one and only one line a. Axiom 4: Any two distinct lines a, b are incident with at least one point B.

#2. pg. 37. Exercise 4.

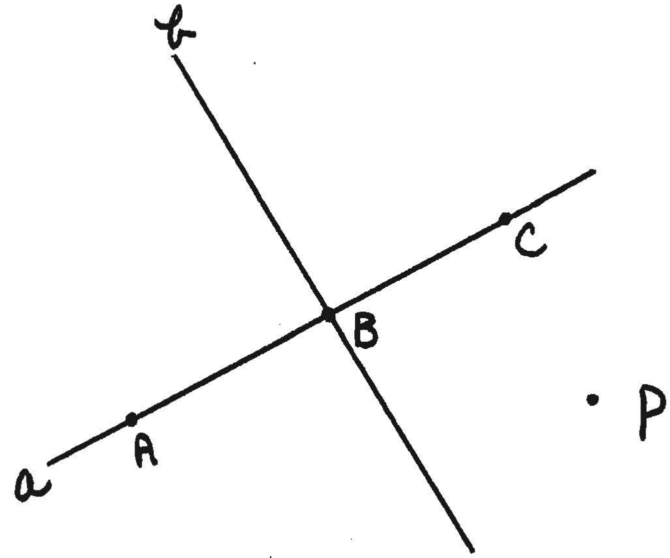
Show that in a projective plane there exist four points A, B, C, D, no three of which are collinear. Refer to drawing 2. and Theorem 3.22 which says each point is incident with at least three lines.

#3. pg 37. Exercise 7.

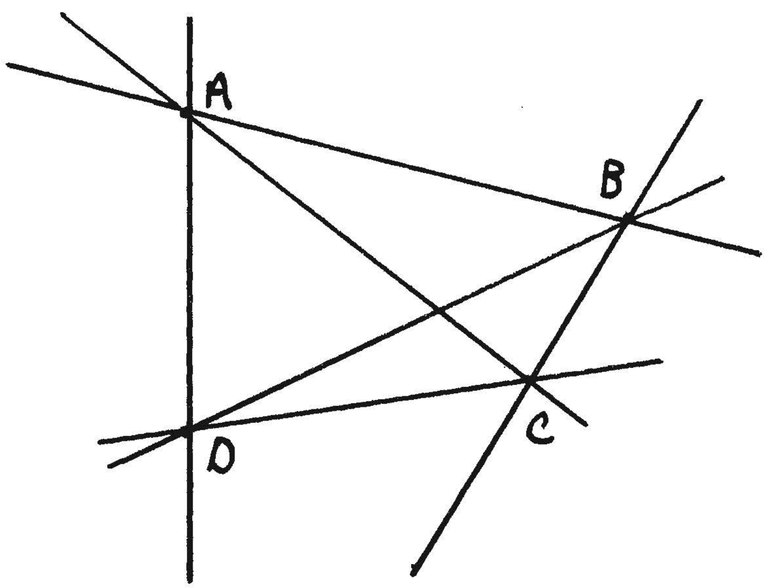
Write duals of the following: (a) Three non-collinear points A,B,C and the three lines a,b,c determined by them. Dual Three nonconcurrent lines a, b, c and the points A,B,C determined by them. Refer to drawing 3. (b) Four points, A, B, C, D, no three of which are collinear, and the six lines determined by them. Dual Four lines a,b,c,d, no three of which are concurrent and the six points A, B, C, D, E, F determined by them. Refer to drawing 4.

#4. pg. 38. Exercise 2.

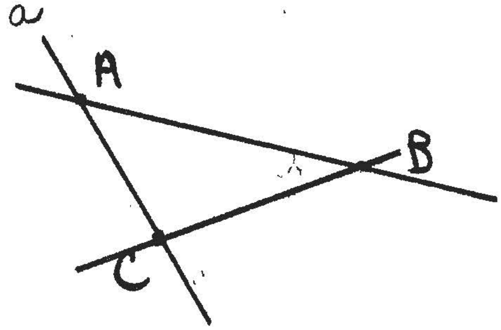
Construct a 13-point finite projective plane and show that axiom 5 is verified. Axiom 5: If two triangles are perspective from a point, then they are perspective from a line. Refer to drawing 5. Triangle BCK and triangle BFH are perspective from C and line BH.



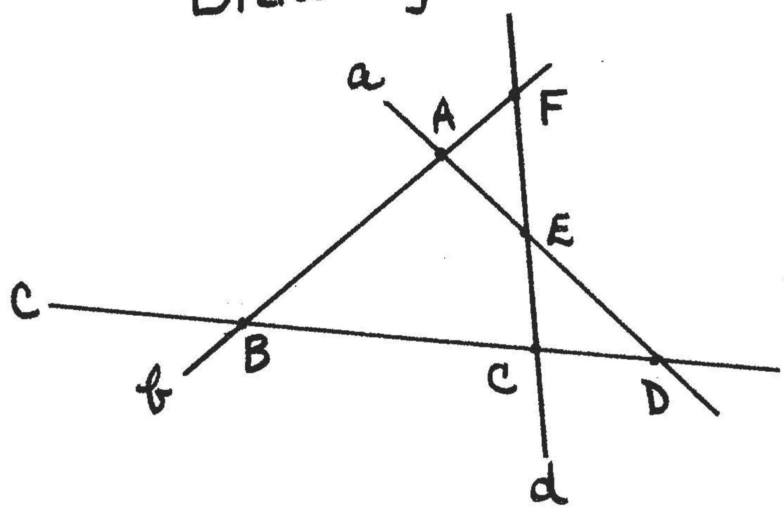
Drawing 1.



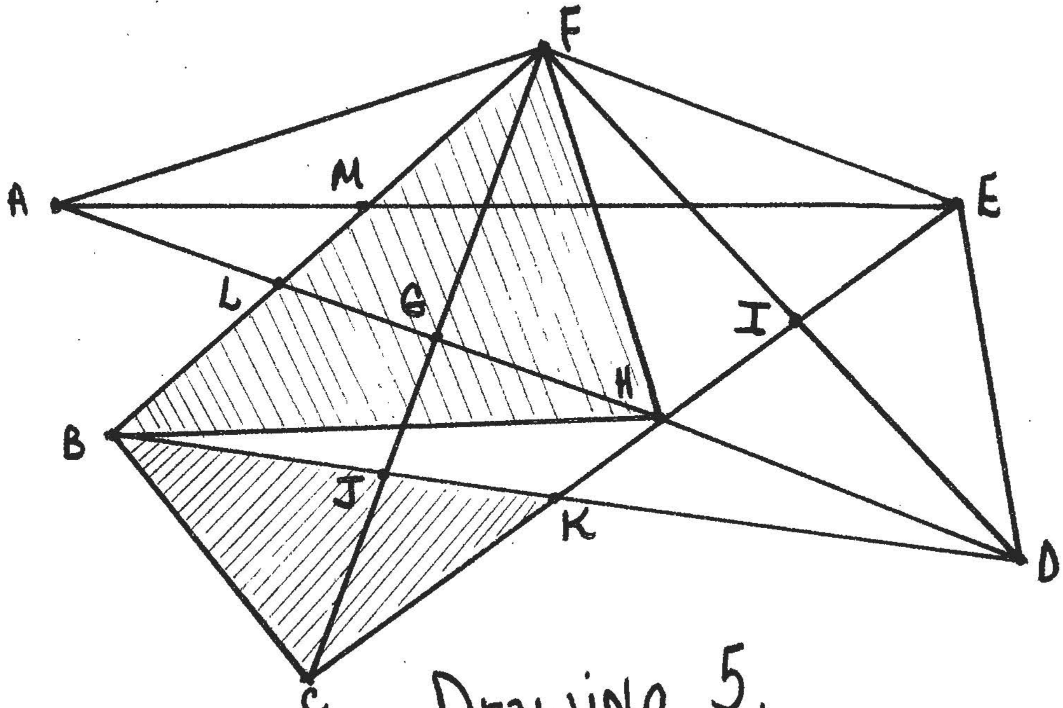
Drawing 2.



Drawing 3.



Drawing 4.



Drawing 5.

#5. pg. 39. Exercise 5.

Justify the following instructions for the construction of a line joining a given point Q to the inaccessible point of intersection P of two given lines a and b.

Let O be an arbitrary point and p_1, p_2, p_3 be three arbitrary lines through O. Let $a \cdot p_1 = A$, $b \cdot p_1 = A'$, $a \cdot p_2 = B$, $b \cdot p_2 = B'$, $AQ \cdot p_3 = C$, $A'Q \cdot p_3 = C'$, and $BC \cdot B'C' = R$. Then the required line is QR. Refer to drawing 6. The triangles A'B'C', and ABC are perspective from O b y construction, and therefore perspective from a line by Axiom 5 or Desargues' Theorem. Therefore, the inaccessible point of intersection P of lines a,b, lies on the line RQ.

#6. pg. 43, Exercise 1.

Show that if ABC is the diagonal triangle of a quadrangle PQRS and if $X = BC \cdot QR$, $Y = CA \cdot RP$, $Z = AB \cdot PQ$, then X,Y,Z are collinear. Refer to drawing 7. Triangles ABC and PQR are perspective from point S, and by Axiom 5 they are perspective from line Z,X,Y.

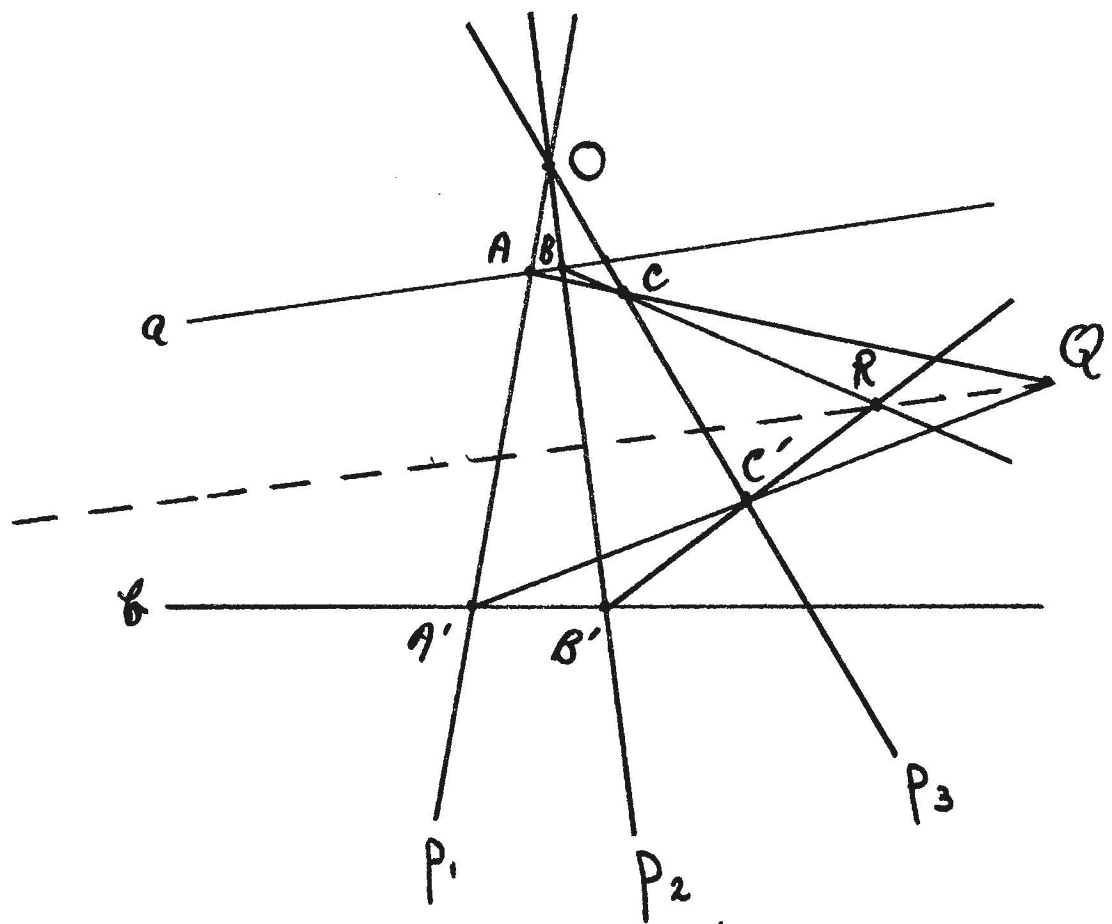
#7. pg. 43. Exercise 2.

State the dual of exercise 1 and make a drawing for each. Show that if ABC is the diagonal triangle of quadrangle PQRS and if $X = BC \cdot QR$, $Y = CA \cdot RP$, $Z = AB \cdot PQ$, then X,Y,Z, are collinear.

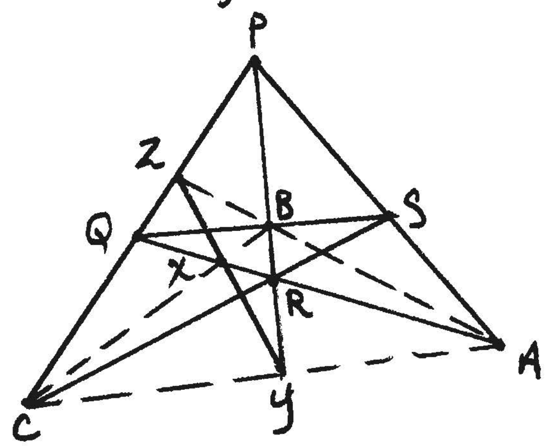
<u>Dual</u> Show that if the join of abc forms the diagonal triangle of quadrilateral pqrs and if $x = bc \cdot qr$, $y = ca \cdot rp$, $z = ab \cdot pq$, then x,y,z are concurrent. Refer to drawings 7 and 8.

#8. pg. 43. Exercise 3.

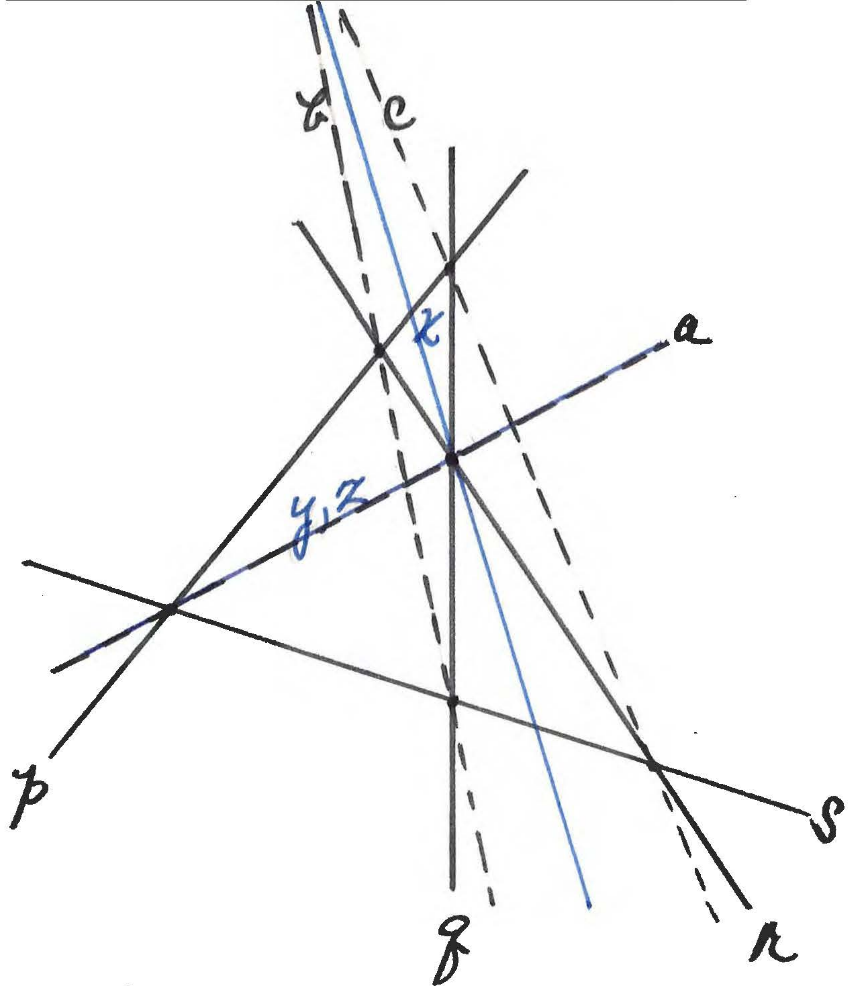
Given the diagonal triangle and one vertex of a quadrangle, construct the complete quadrangle. Is it unique? Refer to drawing 9. Draw a triangle and a point anywhere. Connect the vertices and the



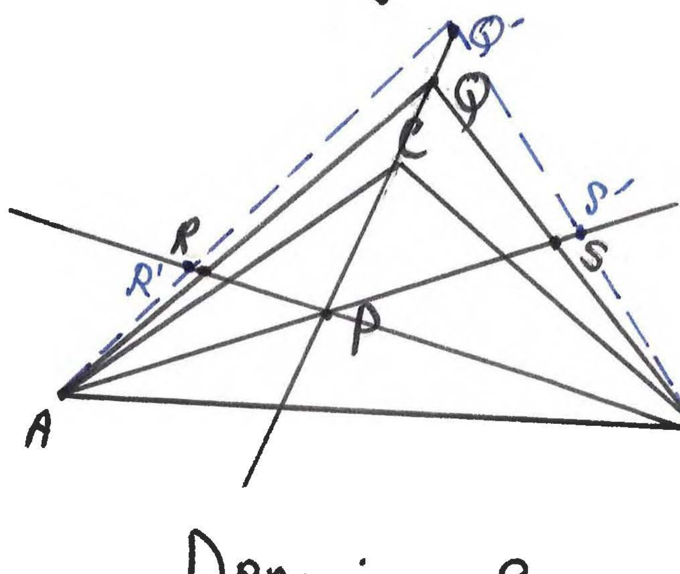
Drawing 6.



Drawing 7.



DRAWING 8.



DRAWING 9.

point with lines. Pick another point on any of these lines and connect it to the vertices. The intersections are the other points. No the quadrangle is not unique.

#9. pg. 43. Exercise 4.

Define a harmonic set of lines and construct a line d such that H(a,b,c,d) when three concurrent lines a,b,c are given.

Four concurrent lines a,b,c,d are said to form a harmonic set if there is a quadrilateral of which two opposite vertices lie on line a, two opposite vertices lie on line b and the remaining vertices lie on c and d respectively. Refer to drawing 10. The vertices are A,B on a, C,D on b, E on c and F on d.

#10. pg 46. Exercise 1.

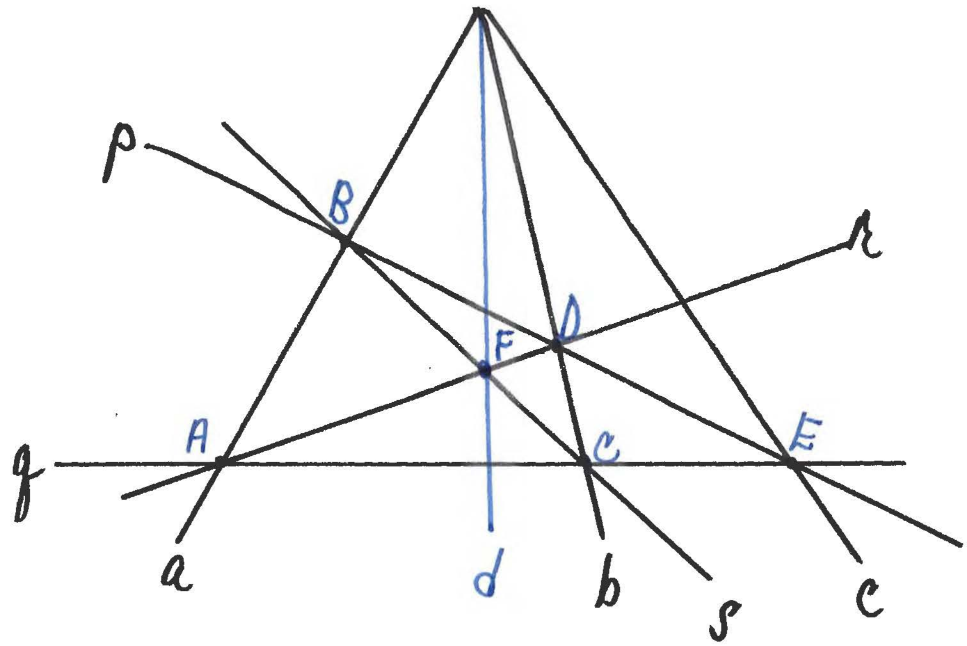
In the projectivity given in drawing 11 show how to find the point X' on p' corresponding to any point X on p. Draw PX and label the intersection on p_1 X_1. Draw X_1Q and the intersection on p' is X'.

#11. Pg. 46. Exercise 2.

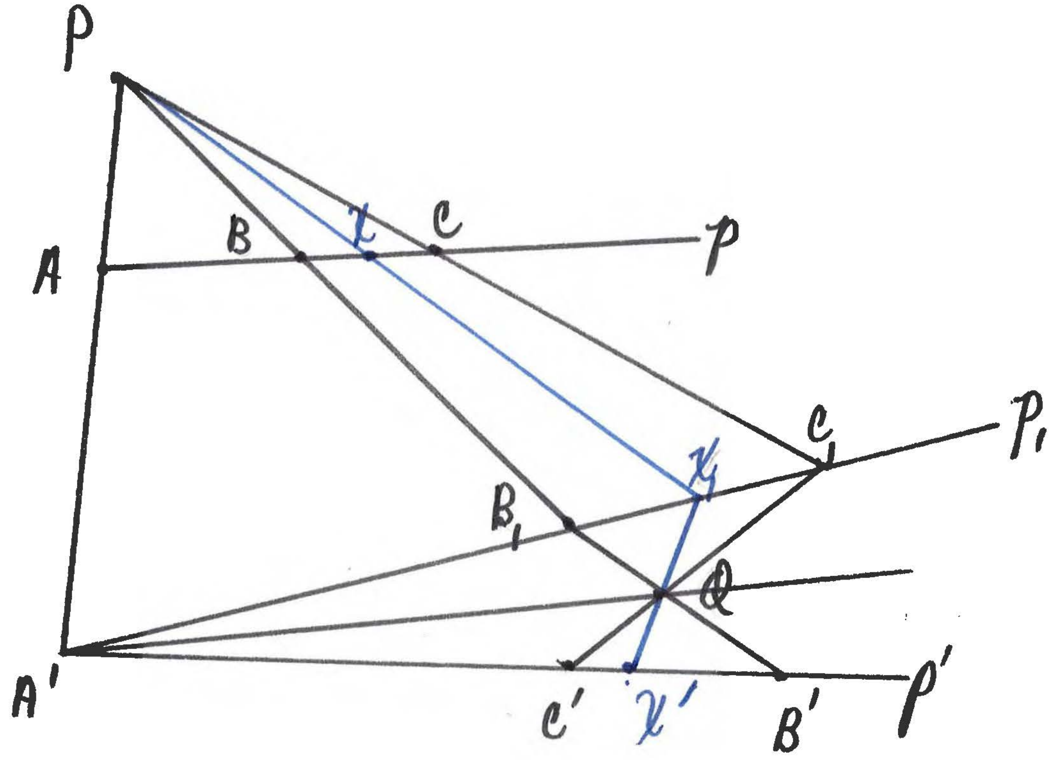
Carry out the construction of Theorem 3.53 for a projectivity between two pencils of lines and for a projectivity between a pencil of points and a pencil of lines. Theorem 3.53: A projectivity may be set up whenever there are three distinct elements of one pencil and the corresponding three elements of another are given. Refer to drawing 12.

#12. pg. 46. Exercise 3.

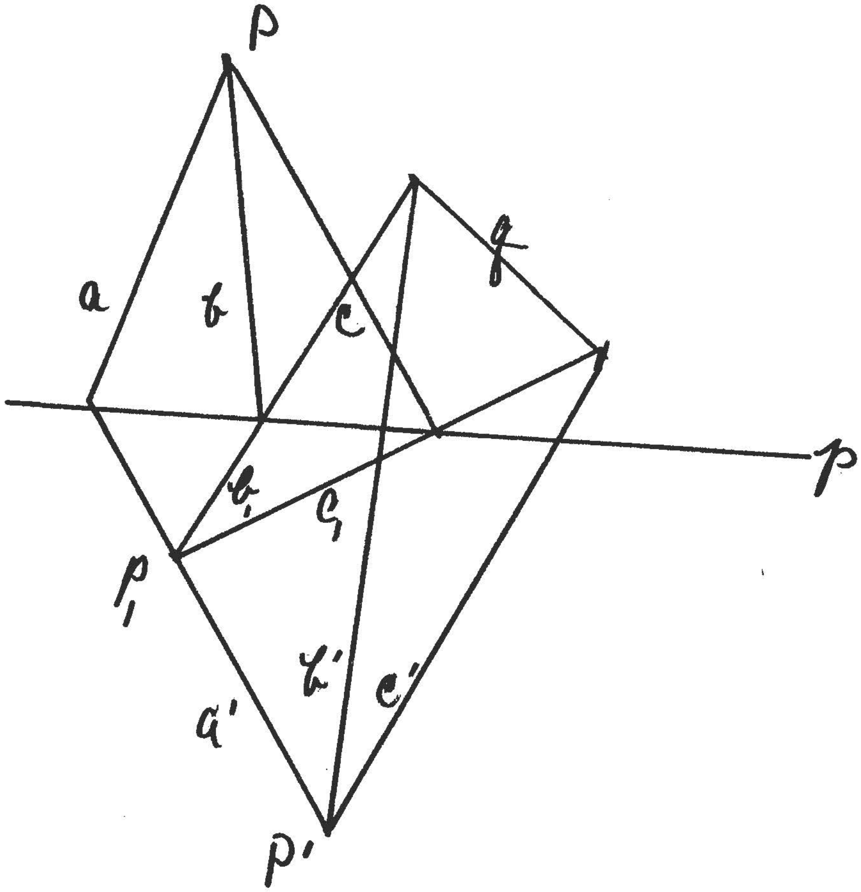
Let a_1, a_2, . . and b_1, b_2, . . be two pencils of lines through the points A and B respectively, such that $a_i \barwedge b_i$. Let p be a line not containing A or B and let $a_i \cdot p = P_i$ and $b_i \cdot p = Q_i$. Show that $P_i \barwedge Q_i$. Refer to drawing 13.



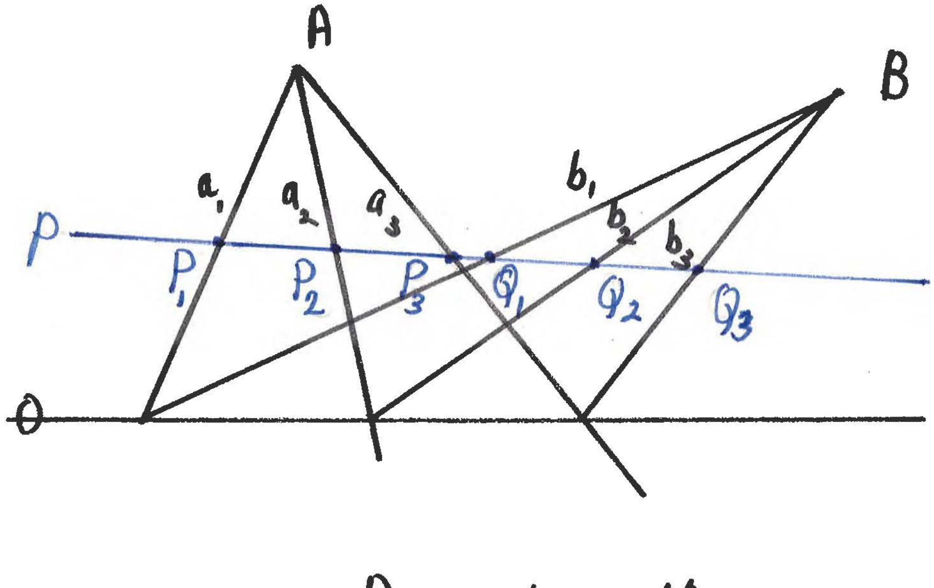
Drawing 10.



Drawing 11.



Drawing 12.



Drawing 13.

$P_1 \overset{A}{\bar{\wedge}} a_1 \overset{\bullet}{\bar{\bar{\wedge}}} b_1 \overset{B}{\bar{\wedge}} Q_1$. $P_1 \bar{\wedge} Q_1$.

#13. pg. 46. Exercise 4.

Let a,b,c be three concurrent lines and P,Q two points not on any of them. Let A_1, A_2 . . and B_1, B_2 . . . be pencils of points on a and b respectively, such that $A_1P \cdot B_1Q = C_1$ where C_1 is on line C. Show that $A_1 \bar{\wedge} B_1$. Refer to drawing 14. $A_1A_2 \overset{P}{\bar{\wedge}} C_1C_2 \overset{Q}{\bar{\wedge}} B_1B_2$. Therefore, $A_1A_2 \bar{\wedge} B_1B_2$.

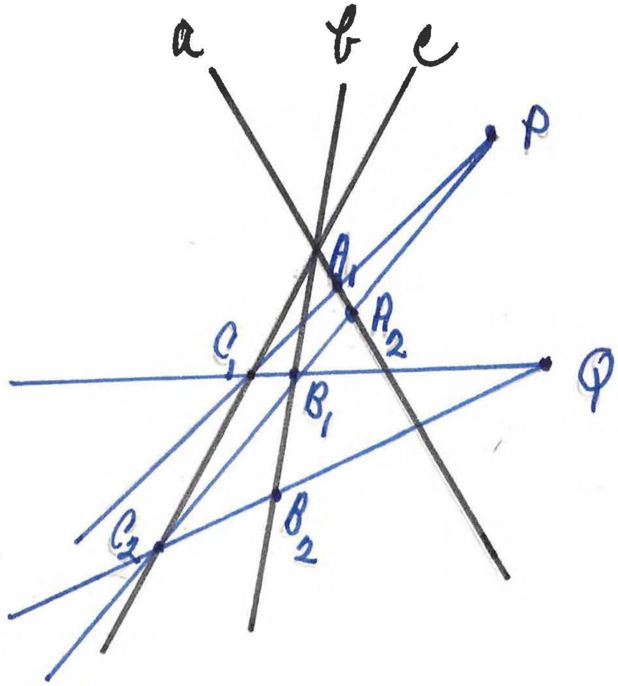
#14. pg. 47. Exercise 15.

Show that ABD $\bar{\wedge}$ BAD $\bar{\wedge}$ DBA. Refer to drawing 15.

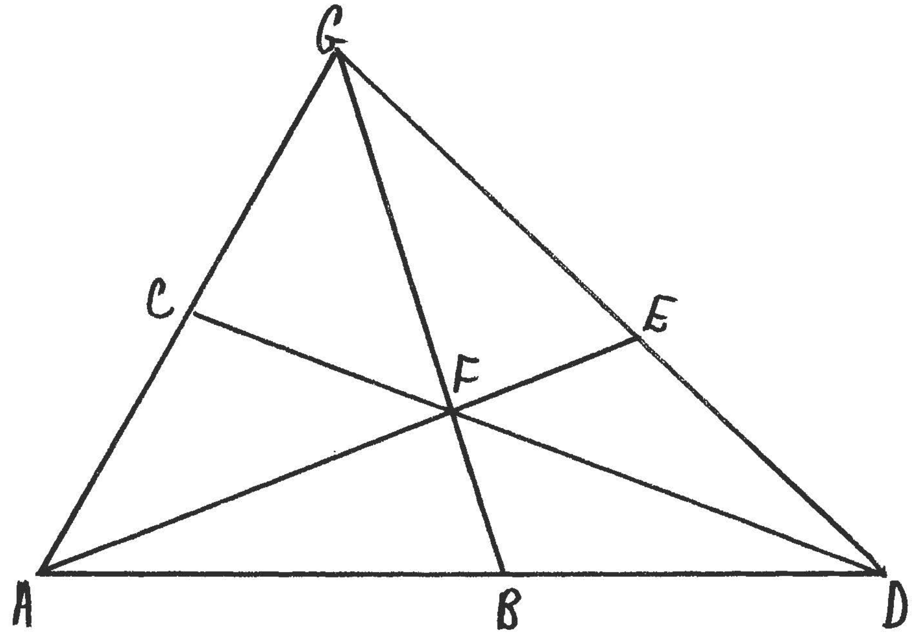
ABD $\overset{G}{\bar{\wedge}}$ DFD $\overset{A}{\bar{\wedge}}$ GED $\overset{F}{\bar{\wedge}}$ BAD, Therefore, ABD $\bar{\wedge}$ BAD.

#15. pg. 53. Exercise 1.

Given 0, 1, ∞ construct 1/2, 1/3, 2/3. Refer to drawing 16. Choose three arbitrary points as reference points on the projective line p. Assign them the values 0 and 1 and the third the symbol ∞. Let q and r be two distinct lines through ∞ other than p. Let A be a point on r. Join A to point 1 and let Q_1 be intersection of this line with q. Join 1 to point 0 and let B be the intersection of this line with r. Join B to point 1 and let Q_2 be the intersection of this line with q. Then the point 2 is the poin $p \cdot AQ_2$. Join B to 2 and let Q_3 be the point of intersection of this line with q. Then point 3 is the point $p \cdot AQ_3$. For -1 join A to 0 and let Q_{-1} be the point of intersection of this line with q. Then point -1 is the point $p \cdot BQ_1$. To find 1/2 construct the H(1,-1,2,1/2) as follows. Given lines AQ_1 and BQ_2 take point Q_1. Let $2Q_2$ intersect AQ_1 at A and $-1Q_1$ intersect AQ_1 at R. Let $-1A \cdot 1Q_1 = S$. Then the required point 1/2 = $RS \cdot p$. To find 1/3 construct H(1,-1,3,1/3). Given 1,-1,3 construct

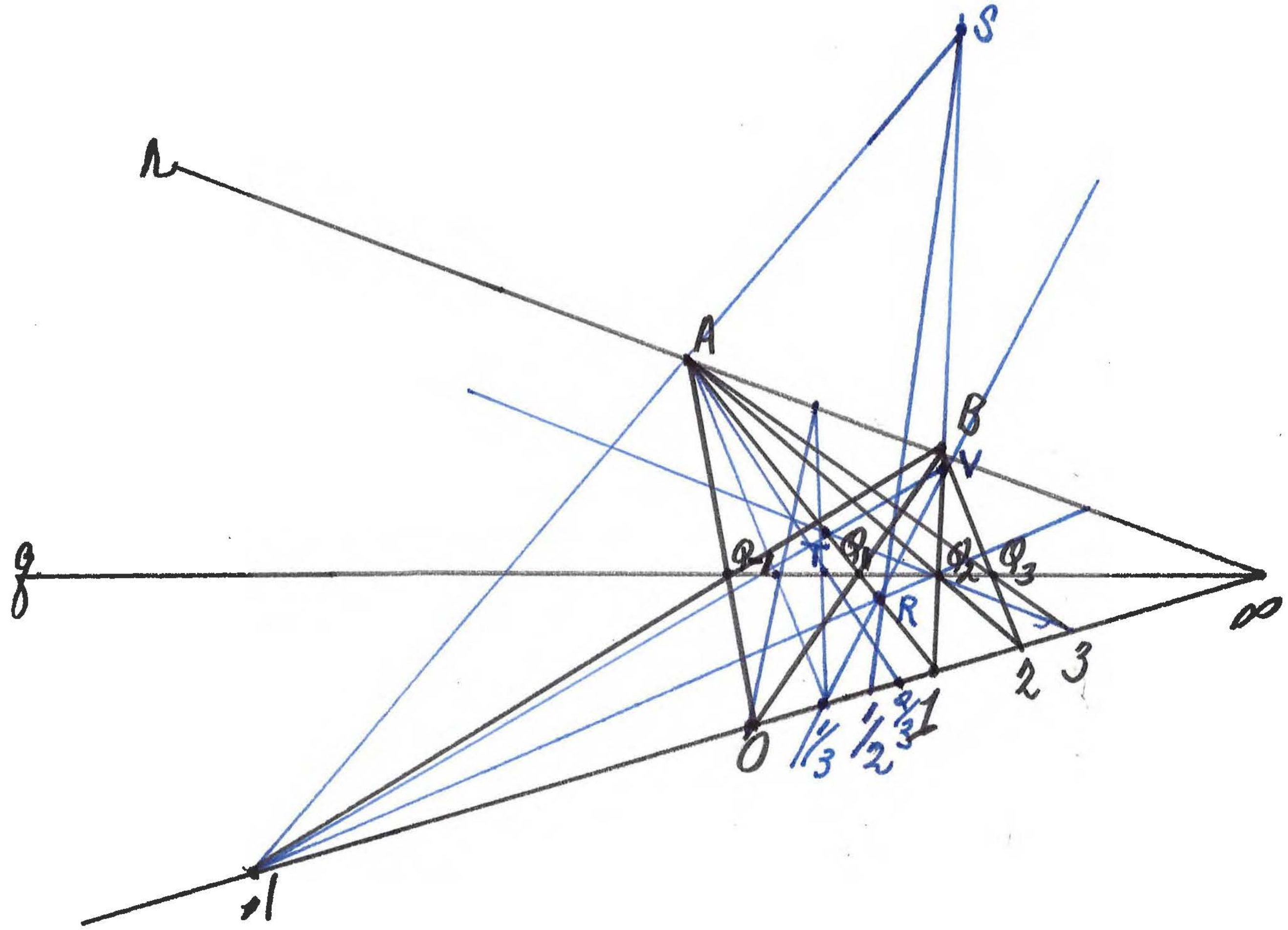


Drawing 14.



Drawing 15.

1/3 as follows. Given AQ_1 and BQ_2 take point Q_1. Let $3Q_2$ intersect AQ_1 at T and $-1Q_1$ intersect AQ_1 at R. Let $-1T \cdot 1Q_1 = M$. Then the required point 1/3 = RM.p. To find 2/3 use the points 0, 1/3, ∞ as the three starting points in place of 0, 1, ∞ and proceed as for point 2 above.



Drawing 16.