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The Regular Polyhedra: A Study in Visual Aids for Teaching Geometry

Sammye Halbert
Ouachita Baptist University

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Traditionally, mathematics, past simple addition, subtraction, multiplication, and division, has been thought of as being so boring, irrelevant, and in short, one of the unavoidable evils of school. An advertisement in "The Mathematics Teacher" expressed the general attitude of many students when it said, "Mathematics was invented by an old magician in the desert who, with the help of his talking monkey, bakes equations and cupcakes in the hot sun." It seems that many students think mathematics is just one problem after another that has some mystical answer floating around in the air somewhere. The object is to get that answer with a minimun of effort and usually with little or no understanding of why it is the correct answer. The only thing that <u>really</u> matters is that right answer.

For the person who has been able to see through this and has taken the time to try and understand a few "whys" about the subject, mathematics can become interesting and well worth the time spent studying it. But, unfortunately, not all students are able to do this, or even want to. Most are just trying to discover the "trick" of how to make the thing work and could not care less why it does.

The tragedy, though, is that, more than likely, it is the teacher or the textbook, not the subject itself, that is the source of its unpopularity. This is not to say that all math teachers are bad teachers or that all math textbooks are bad books. It just seems alot of teachers and authors go about their jobs with about as much motivation as many students have in the classroom. There are, of course, many exceptions to the rule. But the fact still remains, quite a few math teachers seem to accept their fate as a dull teacher, teaching a dull subject, never once looking to new and intriguing ideas to improve their classroom techniques.

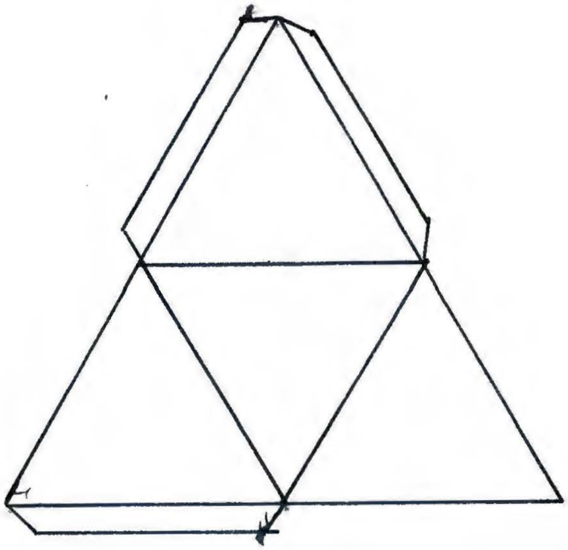
Science, English and history classes have been using techniques and gimmies, which is actually what they are, for years. But mathematics has been slow to catch on. There is nothing wrong with a gimmic, if it is used as a teaching aid and not a substitute for teaching. Bright colores, pictures, and diagrams in textbooks and models,games and other devices in the classroom are good to attract attention, and often this is all that is needed to get students started. But they should illustrate the point to be made and not used just as busy work to keep a class quiet.

One of the most useful devices for mathematics is visual aids in the form of models. This is especially true in geometry classes where the abstractness of the subject may elude a student who does not form mental pictures easily. The following pages are devoted to the construction, use and history of the five regular polyhedra for a mathematics class in the secondary school.

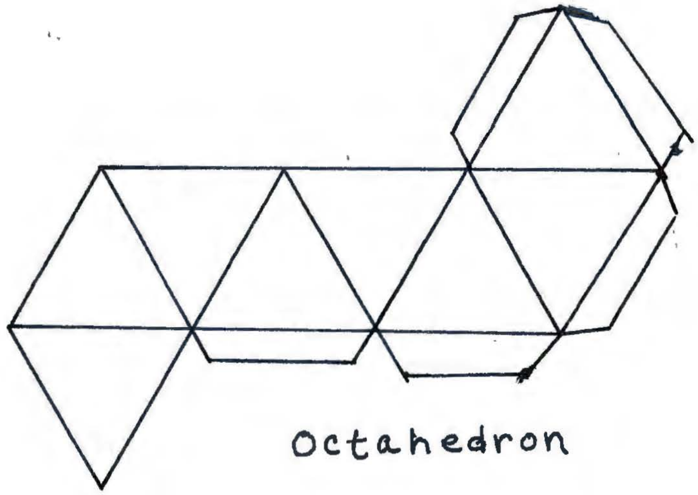
A good way to begin a discussion of the five regular polyhedra would be to first present the solids to the class for observation. Instead of pointing out that each one has identical faces, ask what is similar about each solid. After they have discovered that the tetrahedrom, octahedron, and icosahedron all have congruent triangles for faces, the hexahedron, or cube, has congruent squares for faces, and the dodecahedron has congruent pentagons for faces, ask them to count the number of faces for each figure. This sounds juvenille, but trying to count the faces on a dodecahedron or on a icosahedron can become difficult. Perhaps, if the class has studied descriptive nameing, they would be able to come up with the correct and appropriate names for each solid. Ask the class to make general statements about the solids and guide them in formulating general assumptions.

The purpose of this is to acquaint the student with the various properties of the regular polyhedra by observation rather than rote memorization of material given by the teacher. This is based on the theory that a person remembers things longer and understands them more fully if he has made some effort in the discovery, rather than just hearing it and accepting it without proof.

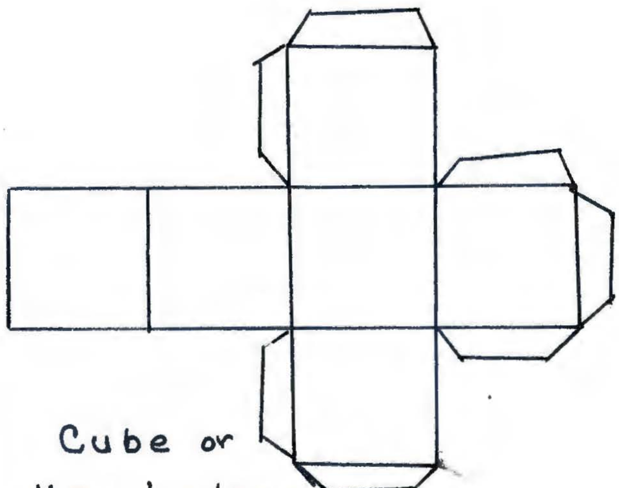
Additional work with the polyhedra may be done by asking how to construct these models. The models accompanying this paper are, of course, much larger than ones that would be constructed in class. because they are large enough for all students to see with some degree of detail. Probably the first problem the students will encounter in their own construction will be making an accurate pattern for the face. Construction of equilateral triangle and squares has probably already been taught, but few geometry classes do much about constructing a regular pentagon. The main and most obvious problem is how large to make the angle between adjacent sides. This is solved by the formula (N-2/N) 180, when N is the number of sides in the polygon. After the construction of the face pattern is done, let them proceed with making an entire pattern. Hopefully, they will turn out similar to the diagram on the next page. If this method is too time consuming or not suited to the class, patterns of the solids may be run off ahead of time.



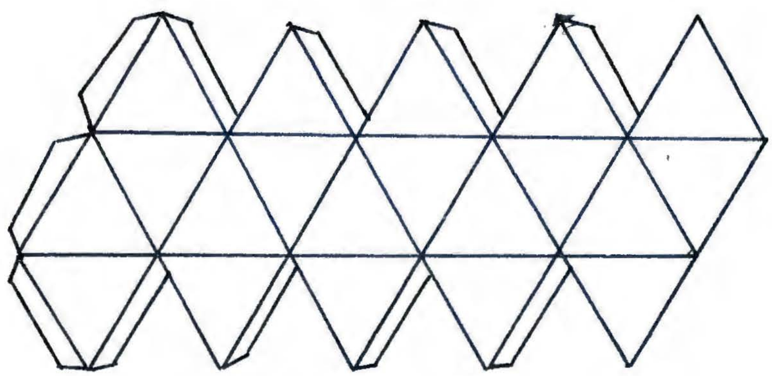
Tetrahedron



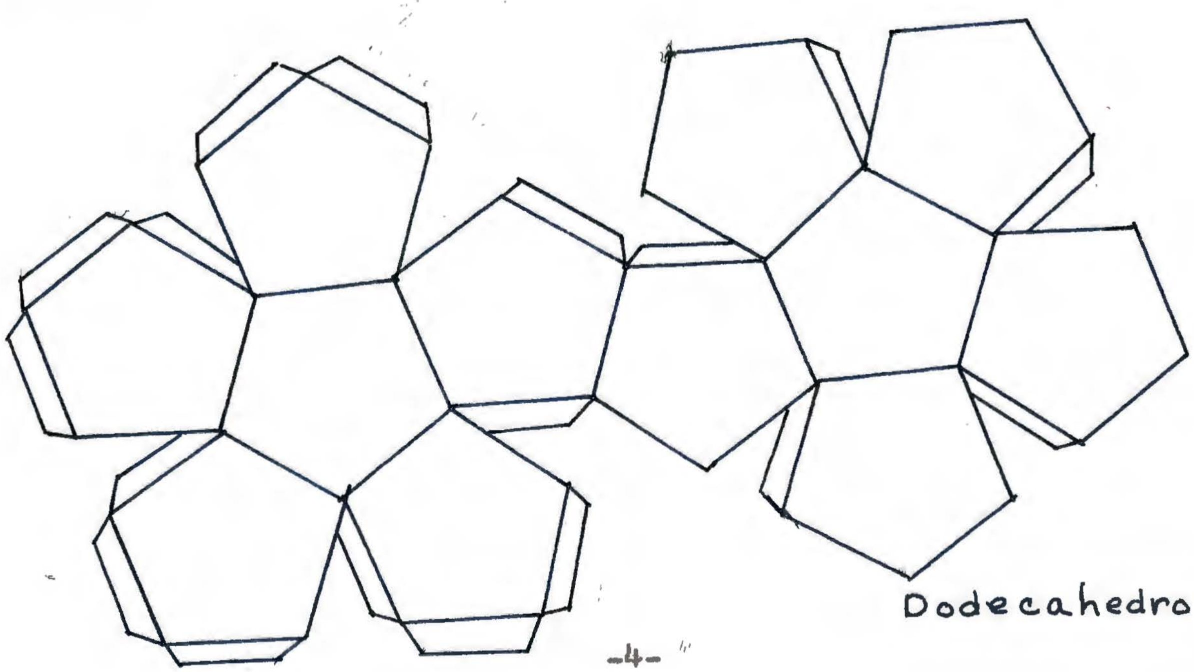
Octahedron



Cube or
Hexahedron



Icosahedron



Dodecahedron

The actual pasting or taping the model together is a worthwhile project. Some students, because they are used to working at a slow pace and are more patient, do quite well in this type of work which demands a degree of patience that must brighter students have hot had to develop. In this way, these students have the chance to enjoy the satisfaction of doing well in class.

After a few models have been made, the teacher can illustrate the idea of a plane passing through each of them by cutting through the side. This will show that a plane through a hexahedron and an octahedron produces a four-sided figure, in the tetrahedron and icosahedron, a five-sided figure. Also illustrate that the figures will have different dimensions as the angle of the plane passing through is changed.

Also stressthe fact that these shapes can be seen in nature and are also man-made. The tetrahedron occurs naturally as a crystal of sodium sulphantimoniate, the cube as a crystal of common salt, and the octahedron as a crystal of chrome alum. The other two, the dedecahedron and the icosahedron, have been observed not as crystals but in the skeletons of tiny sea animals called radiolaria. They are seen as man-made products in architecture and recently the dodechedron has been seen as a desk calendar, because of its twelve faces.

The history of four of the regular polyhedra, the tetrahedron, hexahedron, octahedron and icosahedron, is vague becuase they originated before historical records were kept. But it is known that the ancient Egyptians knew of them because some were used in their architecture. It is thought that the Pythagoreans originated three of the regular polyhedra around 500 B.C. Tradition gives credit to Hippasus, a Pythagorean of the 5th Century B.C., for devising the fifth of the regular polyhedra, the dodecahedron. A story tells that because he took credit for making an addition to the perfect solids given to man by the gods, he drowned at sea.

Later, in the 4th Century B.C., Plato writes of the five solids in Timaeus. In this work, Plato associated four of the five solids with the Empedoclean primal elements of all material bodies - fire, air, water and earth. The fact that there were five solids and only four elements did not hinder Plato's theory. He explained the fifth one, the dodecahedron, by associating it with the all encompassing sphere of the universe.

Plato explained his associations like this. Since the earth is stable and immovable, it should be represented by the most stable of the solids, the cube or hexahedron. Water is represented by the icosahdron because water is harder to move than air or fire, and since the icosahedron has the most faces, it would seem harder to move. Fire is made of small and acute bodies, and is sharp and cutting, therefore it is represented by the tetrahedron. Finally, air is composed of octahedron solids. His theory that the dodecahedron represented the universe may have come from the fact that its volume is the closest of the five to the volume of the sphere in which they are inscribed. Because of all the work and study Plato devoted to the five regular polyhedra, they are often referred to as the Platonic solids. Euchid also studied the Platonic solids to a great degree. In his series of thirteen books, Elements. he begins with the construction of an equilateral triangle and concludes with the five solids. The last proposition in Elements is that no other solids with congruent regular polygons as faces are possible, and he proceeds to give proof for this proposition.

Johannes Képler, an astronomer and mathematician in the late 16th Century, did further study of the Platonic solids. He assumed, based on Plato's Timaeus, that since the tetrahedron had the smallest volume for its surface it should represent dryness and the icosahedron which he believed (incorrectly) to enclose the largest volume, represented wetness, because the volume-surfaces relation is also a quality of dryness and wetness. Therefore, fire being the driest, it was the tetrahedron, and water, being the wettest, it was the icosahedron. The cube was associated with the earth because of its stability and the octahedron was associated with air because held at two opposite vertices, it spins freely and therefore has the instability of air. The dodecahedron was associated with the universe because it has twelve faces and the zodiac has twelve signs.

Kepler even went further to say that the five solids accounted for the number of planets (five were know at the time) and their spacing around the sun. In his Mysterium Cosmographicum of 1596, he says:

> "The orbit of the Earth is a circle: round the sphere to which this circle belongs, describe a dodecshedron; the sphere including this will give the orbit of Mars. Round Mars describe a tetrahedron; the circle including this will be the orbit of Jupiter. Describe a circle areound Jupiter's orbil, the circle including this will be the orbit of Saturn. Now inscribe in the earth's orbit an icoshedron; the circle inscribed in it will be the orbit of Venus; the circle inscribed in it will be Murcury's orbit. This is the reason of the number of planets."[1]

Kepler also was one of the first to study the small and great stellated dodecahedron. They are not convex solids, and therefore not classified as the other 5 solids are, but they do have regular polygon faces. Louis Poinsott, a French mathematician in the late 18th Century, added the two last regular polyhedra, the great dodecahedron and the great icosahedron which also were not convex. This makes a total of 9 regular polyhedra. This still leaves the number of regular solids at five, since they are defined as being regular <u>convex</u> polyhedra.

Ludwig Schlaffi (1814-95), a Swiss mathematician, originated the symbol used now for the regular polyhedra (P,q), where p is the number of sides and q is the number of polygons meeting at each ventrex. His method of proving there are only five was this:

> "Let (P,q) be any regular polyhedra. The size (in degrees) of each angle of the regular polygons forming its sides can be expressed as 180-(360/p). Since (p,q) is convex, the sum of the angles at one ventex is less than 360. Therefore, we can set up the following inequality:
>
> $$(180-{}^{360}/p)q < 360$$
> $$180(1-{}^{2}/p)q < 360$$
> $$(p-2)(q-2) < 4$$
>
> p and q are both > 2. If p=3, we can have from the inequality (3,3), (3,4), (3,5). If p=4, we have (4,3), and if p=5, we have (5,3). Since there is no allowable value for q where $p > 5$, there are no other regular polyhedra."[2]

In concluding, one point stands out from all the rest that has been said. Mathematics can be interesting and enjoyable if you take the time to make it so. A little research and study can make it meaningful and worthwhile. But it is obvious that, like so many other things, teachers and students alike get from it what they put into it.

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(4) Mathematics Teacher, "The Geometry Capsule Concerning the Five Platonic Lolids", Howard Eves, January 1969, pp. 42-43.

(5) Mathematics Teacher, "The World of Polyhedra", March 1965, pp. 244-48.

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FOOTNOTES

[1]The Mathematics Teacher, "The Geometry Capsule Concerning the Five Platonic solids", January 1969.

[2]Historical Topics6for the Mathematics Classroom, NCTM Yearbook, 31st edition, pp. 220-21.