Vectors: A Study of Vector Analysis by H. B. Phillips

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1. $A, B$ are vectors forming consecutive sides of a parallelogram. Find the vectors forming the other two sides.

$$-A$$
$$-B$$

2. $A, B$ are vectors forming consecutive sides of a regular hexagon. Find the vectors forming the other four sides.

$$B-A$$
$$-A$$
$$-B$$
$$A-B$$

3.5. The vectors $A, B, C$ form coterminal edges of a parallelepiped. Show that $A + B + C$ is equal to a vector diagonal.
6. If \( A, B \) are vectors from an origin \( O \) to the points \( A, B \), find the vector \( C \) from \( O \) to the middle point of \( AB \).

\[
\frac{BC}{BA} = \frac{1}{2}
\]

\[
\frac{C-B}{A-B} = \frac{1}{2}
\]

\[
C-B = \frac{1}{2} (A-B)
\]

\[
C = \frac{1}{2} (A-B)+B
\]

\[
C = \frac{1}{2} A - \frac{1}{2} B + B
\]

\[
C = \frac{1}{2} A + \frac{1}{2} B
\]

\[
C = \frac{1}{2} (A+B)
\]

8. From the center of a regular pentagon vectors are drawn to its vertices. Show that their sum is zero.

\[
\begin{align*}
\vec{F} &= 1 \vec{F}_x + 0 \vec{F}_y \\
\vec{G} &= 0.30902 \vec{G}_x + 0.95106 \vec{G}_y \\
\vec{H} &= -0.80902 \vec{H}_x + 0.58779 \vec{H}_y \\
\vec{I} &= -0.80902 \vec{I}_x + 0.58779 \vec{I}_y \\
\vec{J} &= 0.30902 \vec{J}_x - 0.95106 \vec{J}_y
\end{align*}
\]

\[
\begin{align*}
F_x + G_x + H_x + I_x + J_x &= 0 \\
F_y + G_y + H_y + I_y + J_y &= 0
\end{align*}
\]

\[
\Rightarrow F + G + H + I + J = 0
\]
9. \[ \frac{DF}{DC} = \frac{1}{2} \]

\[ \frac{CE}{CB} = \frac{1}{2} \]

**AF, DH Medians \( \triangle ACD \)**

\[ \frac{HG}{GD} = \frac{1}{2} \]

\[ HG = \frac{1}{3} HD \]

\[ 3HG = HD \]

**BH, AE Medians \( \triangle ABC \)**

\[ \frac{HI}{IB} = \frac{1}{2} \]

\[ HI = \frac{1}{3} HB \]

\[ HG + HI = GI \]

\[ GI = \frac{1}{3} BD \]

\[ HD = HB \]

\[ HI = \frac{1}{3} HD \]

\[ BD = BH + HD \]

\[ GD = \frac{2}{3} HD \]

\[ BD = HD + HD \]

\[ IB = \frac{2}{3} HB \]

\[ BD = 2HD \]

\[ BD = 2(3HG) \]

\[ GD = IB \]

\[ GD + IB = \frac{2}{3} BD \]

\[ BD = 6HG \]

\[ \frac{1}{6} BD = HG \]

\[ \frac{1}{6} BD = HI \]

\[ \therefore GD = \frac{1}{3} BD \]

\[ IB = \frac{1}{3} BD \]
13. The vectors \( A = i + j \) and \( B = 3i - 2j \) extend from the origin of coordinates. Show that the line joining their ends is parallel to the X-Y plane and find its length.

\[
B - A = 2i - 3j
\]

\[
|B - A| = \sqrt{2^2 + (-3)^2} = \sqrt{13}
\]

The line joining their ends is parallel to the X-Y plane and one unit below it. Its length is \( \sqrt{13} \).

14. The vectors from the origin to the points \( A, B, C, D \) are \( A = i + j + k \),

\[
B = 2i + 3j,
\]

\[
C = 3i + 5j - 2k,
\]

\[
D = k - j
\]

Show that the lines \( AB \) and \( CD \) are parallel and find the ratio of their lengths.

\[
B - A = 2i - 3j - k
\]

\[
\frac{AB}{|B - A|} = \frac{1D - C|}{\sqrt{13}}
\]

\[
D - C = -3i - 6j + 3k
\]

\[
\frac{AB}{\sqrt{13}} = \frac{\sqrt{1^2 + 2^2 + (-1)^2}}{\sqrt{13}} = \frac{1}{\sqrt{13}}
\]

\[
D - C = -3(B - A)
\]

\[
\frac{CD}{AB} = \frac{3N6}{N6} = 3
\]

\[
CD = 3N6
\]
15. Show that the vectors $i - j$, $j - k$, $k - i$ are parallel to a plane.

\[
A = i - j \\
B = j - k \\
C = k - i
\]

\[
AXB = [(1)(-1) - (0)(1)]i + [(0)(0) - (1)(-1)]j + [(0)(1) - (-1)(0)]k
\]

\[
AXB = i + j + k
\]

\[
C \cdot (AXB) = -1 + 1 = 0
\]

\[
\therefore \cos \theta = 0 \\
\theta = 90^\circ
\]

\[
\therefore C \perp AXB \\
\therefore C \parallel \text{the plane of } A' \text{ and } B
\]
18. Show that the vectors

\[ A = 2i - j + k \]
\[ B = 1 - 3j - 5k \]
\[ C = 3i - 4j - 4k \]

form the sides of a right triangle.

\[ AB = \sqrt{(1-2)^2 + (-3+1)^2 + (-5-1)^2} \]
\[ AB = \sqrt{1 + 4 + 36} \]
\[ AB = \sqrt{41} \]

\[ BC = \sqrt{(3-1)^2 + (-4+3)^2 + (-4+5)^2} \]
\[ BC = \sqrt{4 + 1 + 1} \]
\[ BC = \sqrt{6} \]

\[ AC = \sqrt{(3-2)^2 + (-4+1)^2 + (-4-1)^2} \]
\[ AC = \sqrt{1 + 9 + 25} \]
\[ AC = \sqrt{35} \]

\[ (AB)^2 = (AC)^2 + (BC)^2 \]
\[ 41 = 35 + 6 \]
\[ 41 = 41 \]

:: Δ ABC is a right triangle
19. Find the cosine of the angle between the two vectors:

\[ \mathbf{A} = 1 \mathbf{i} - 2 \mathbf{j} - 2 \mathbf{k} \]
\[ \mathbf{B} = 2 \mathbf{i} + \mathbf{j} - 2 \mathbf{k} \]

\[ \mathbf{A} \cdot \mathbf{B} = \|\mathbf{A}\| \|\mathbf{B}\| \cos \theta \]
\[ 2 - 2 + 4 = \sqrt{1 + 4 + 4} \cdot \sqrt{4 + 1 + 4} \cdot \cos \theta \]
\[ 4 = \frac{\sqrt{9} \cdot \sqrt{9}}{} \cdot \cos \theta \]
\[ 4 = \frac{9 \cos \theta}{9} \]
\[ \frac{4}{9} = \cos \theta \]
\[ .4444 = \cos \theta \]
21. Prove that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its four sides.

\[(AC)^2 + (DB)^2 = (DA)^2 + (AB)^2 + (CB)^2 + (DC)^2\]

\[(DB)^2 = (DC)^2 + (CB)^2 - 2(DC)(CB) \cos \angle DCB\]

\[(AC)^2 = (AB)^2 + (CB)^2 - 2(AB)(CB) \cos \angle ABC\]

\[\angle ABC = 180^\circ - \angle DCB\]

\[\cos \angle ABC = -\cos \angle DCB\]

\[AB = DC\]

\[CB = DA\]

\[(AC)^2 = (DC)^2 + (CB)^2 - 2(DC)(CB)(- \cos \angle DCB)\]

\[(AC)^2 = (DC)^2 + (CB)^2 + 2(DC)(CB) \cos \angle DCB\]

\[(AC)^2 + (DB)^2 = (AB)^2 + (DC)^2 + (CB)^2 + (DA)^2\]
22. By squaring both sides of the equation
   \[ A = B - C \]
   and interpreting the result geometrically, prove the formula
   \[ A^2 = B^2 + C^2 - 2BC \cos \alpha. \]

   \[ A = B - C \]
   \[ |A|^2 = |B|^2 - 2B \cdot C + |C|^2 \]

   \[ B \cdot C = |B| \cdot |C| \cos \alpha \]

   \[ A^2 = B^2 - 2BC \cos \alpha + C^2 \]

   \[ \boxed{A^2 = B^2 + C^2 - 2BC \cos \alpha} \]
23. Show that $A = i \cos \alpha + j \sin \alpha$, and $B = i \cos \beta + j \sin \beta$ are unit vectors in the $XY$-plane making angles $\alpha$, $\beta$ with the $X$-axis. By means of the scalar product obtain the formula for $\cos (\alpha - \beta)$.

\[
\begin{align*}
\frac{i \cos \alpha}{A} &= \cos \alpha & \frac{i \cos \beta}{B} &= \cos \beta \\
\frac{i \sin \alpha}{A} &= \sin \alpha & \frac{i \sin \beta}{B} &= \sin \beta \\
\frac{j}{i} &= A \sin \alpha & \frac{j}{i} &= B \sin \beta
\end{align*}
\]

\[
\begin{align*}
\theta &= \alpha - \beta \\
A \cdot B &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\
|A| |B| \cos \theta &= \frac{\sqrt{\cos^2 \alpha + \sin^2 \alpha}}{\sqrt{\cos^2 \beta + \sin^2 \beta}} \\
&= \frac{\cos (\alpha - \beta)}{\sqrt{\cos^2 \alpha + \sin^2 \alpha}} \cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \\
\cos^2 + \sin^2 &= 1
\end{align*}
\]

\[\therefore \cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta\]
27. Given

\[ A = i - 2j, \quad B = i + k \]

find the component of \( A \) along \( B \).

\[ a = |A| \cos \theta \quad \text{is component of } A \text{ along } B. \]

\[ A - B = |A| |B| \cos \theta \quad |B| = \sqrt{1^2 + 1^2} \]

\[ \frac{A - B}{|B|} = |A| \cos \theta \quad |B| = \sqrt{2} \]

\[ \frac{-2}{\sqrt{2}} = |A| \cos \theta \quad A - B = (1)(0) + (-2)(1) + (0)(0) \]

\[ A - B = -2 \]

\[ -\sqrt{2} = |A| \cos \theta = a \]
29. Given
\[ A = \frac{1}{2} (2i + 3j + 6k) \]
\[ B = \frac{1}{2} (3i - 6j + 2k) \]
\[ C = \frac{1}{2} (6i + 2j - 3k) \]

Show that \( A, B, C \) are of unit length, mutually perpendicular, and that \( AXB = C \).

\[ |A| = \sqrt{\left(\frac{2}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{6}{2}\right)^2} \]
\[ |A| = \sqrt{\frac{4}{4} + \frac{9}{4} + \frac{36}{4}} \]
\[ |A| = \sqrt{\frac{49}{49}} \]
\[ |A| = 1 \]

\[ |B| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(-\frac{6}{2}\right)^2 + \left(\frac{2}{2}\right)^2} \]
\[ |B| = \sqrt{\frac{9}{4} + \frac{36}{4} + \frac{4}{4}} \]
\[ |B| = \sqrt{\frac{49}{49}} \]
\[ |B| = 1 \]

\[ |C| = \sqrt{\left(\frac{6}{2}\right)^2 + \left(\frac{2}{2}\right)^2 + \left(-\frac{3}{2}\right)^2} \]
\[ |C| = \sqrt{\frac{36}{4} + \frac{4}{4} + \frac{9}{4}} \]
\[ |C| = \sqrt{\frac{49}{49}} \]
\[ |C| = 1 \]
29. (Continued)

\[ A \cdot B = |A| |B| \cos \theta \]
\[ \left(\frac{2}{7}\right)\left(\frac{3}{7}\right) + \left(\frac{3}{7}\right)\left(-\frac{6}{7}\right) + \left(\frac{6}{7}\right)\left(\frac{2}{7}\right) = \left(\frac{1}{1}\right) \cos \theta \]
\[ \frac{6}{49} - \frac{18}{49} + \frac{12}{49} = \left(\frac{1}{1}\right) \cos \theta \]
\[ 0 = \cos \theta \]
\[ 90^\circ = \theta \]
\[ \therefore A \perp B \]

\[ B \cdot C = |B| |C| \cos \theta \]
\[ \left(\frac{3}{7}\right)\left(\frac{6}{7}\right) + \left(-\frac{6}{7}\right)\left(\frac{2}{7}\right) + \left(\frac{2}{7}\right)\left(-\frac{3}{7}\right) = \left(\frac{1}{1}\right) \cos \theta \]
\[ \frac{18}{49} - \frac{12}{49} - \frac{6}{49} = \cos \theta \]
\[ 0 = \cos \theta \]
\[ 90^\circ = \theta \]
\[ \therefore B \perp C \]

\[ A \cdot C = |A| |C| \cos \theta \]
\[ \left(\frac{2}{7}\right)\left(\frac{6}{7}\right) + \left(\frac{3}{7}\right)\left(\frac{2}{7}\right) + \left(\frac{6}{7}\right)\left(-\frac{3}{7}\right) = \left(\frac{1}{1}\right) \cos \theta \]
\[ \frac{12}{49} + \frac{6}{49} - \frac{18}{49} = \cos \theta \]
\[ AXB = \frac{\theta}{C} \quad |C| = 1 \]
\[ 0 = \cos \theta \]
\[ 1 = 1 \]
\[ AXB \text{ is a vector} \perp \text{ both} \]
\[ \therefore A \perp C \]

And B, with magnitude of \(|A| |B| \sin \theta = 1\)
30. Find the area of the parallelogram formed by the two vectors

\[ A = 3\mathbf{i} + 2\mathbf{j}, \quad B = 2\mathbf{j} - 4\mathbf{k} \]

Area = \left| \mathbf{A} \times \mathbf{B} \right|
Area = \left| -8\mathbf{i} + 12\mathbf{j} + 6\mathbf{k} \right|
Area = \sqrt{64 + 144 + 36}
Area = \sqrt{244}
\left( \text{Area} \approx 15.62 \right)

31. Find the area of the triangle formed by the points
P_1 (1, 1, 1), P_2 (1, 2, 3), P_3 (2, 3, 1)

\[ \overrightarrow{P_1 P_2} (0, 1, 2) \]
\[ \overrightarrow{P_2 P_3} (1, 1, -2) \]

\[ \text{Area} = \frac{1}{2} \left| \overrightarrow{P_2 P_3} \times \overrightarrow{P_1 P_2} \right| \]
\[ \text{Area} = \frac{1}{2} \left| 4, -2, 11 \right| \]
\[ \text{Area} = \frac{\sqrt{16 + 4 + 1}}{2} \]
\[ \text{Area} = \frac{\sqrt{21}}{2} \]
32. If \( \mathbf{U}_1 \) and \( \mathbf{U}_2 \) are vectors of unit length and \( \theta \) is the angle between them, show that

\[
\sin \frac{1}{2} \theta = \frac{1}{2} |\mathbf{U}_2 - \mathbf{U}_1|.
\]

\[
\frac{DC}{DB} = \frac{1}{2}
\]

\[
\frac{AC - \mathbf{U}_1}{\mathbf{U}_2 - \mathbf{U}_1} = \frac{1}{2}
\]

\[AC - \mathbf{U}_1 = \frac{1}{2} (\mathbf{U}_2 - \mathbf{U}_1)
\]

\[DC = \frac{1}{2} (\mathbf{U}_2 - \mathbf{U}_1)
\]

\[DC = CB
\]

\[CB = \frac{1}{2} (\mathbf{U}_2 - \mathbf{U}_1)
\]

\[
\frac{CB}{\mathbf{U}_2} = \sin \frac{1}{2} \theta
\]

\[CB = \mathbf{U}_2 \sin \frac{1}{2} \theta \quad \mathbf{U}_2 = 1
\]

\[
\frac{1}{2} |\mathbf{U}_2 - \mathbf{U}_1| = \sin \frac{1}{2} \theta
\]
33. The vectors from the origin to the points A, B, C are

\[ A = \hat{i} + \hat{j} - 2\hat{k} \]
\[ B = 2\hat{i} - \hat{j} + \hat{k} \]
\[ C = \hat{i} + 3\hat{j} - \hat{k} \]

Find a vector \( N \) perpendicular to the plane ABC. By projecting \( A \) upon \( N \) find the distance from the origin to the plane.

\[ A - B = -\hat{i} + 2\hat{j} - 3\hat{k} \]
\[ A - C = 0 - 2\hat{j} - \hat{k} \]

\[ (A - B) \times (A - C) = -8\hat{i} - \hat{j} + 2\hat{k} = N \]

\[ A \cdot N = -8 - 1 - 4 = |A| |N| \cos \theta \]
\[ A \cdot N = -13 = |A| |N| \cos \theta \]

\[ \frac{A \cdot N}{|N|} = \frac{|A| \cos \theta}{|N|} \]

\[ \frac{-13}{\sqrt{69}} = \frac{|A| \cos \theta}{|N|} \]
34. In problem 33 find the vector which is the projection of \( A \) upon the plane \( ABC \)

Use vector \( N' = -N = 8\mathbf{i} + \mathbf{j} - 2\mathbf{k} \)

\[
\begin{align*}
A &= \mathbf{i} + \mathbf{j} - 2\mathbf{k} \\
B &= 2\mathbf{i} - \mathbf{j} + \mathbf{k} \\
C &= \mathbf{i} + 3\mathbf{j} - \mathbf{k}
\end{align*}
\]

The distance from the origin to the plane is the length of \( N'' \) and is \( \frac{13}{\sqrt{69}} \).

\( N'' \) is proportional to \( N' \)

\( \therefore N'' = 8K\mathbf{i} + K\mathbf{j} - 2K\mathbf{k} \quad K \text{-constant} \)

\[
N'' = K(8\mathbf{i} + \mathbf{j} - 2\mathbf{k})
\]

\[
\frac{\left| N'' \right|}{N''} = \frac{13}{\sqrt{69}} = \sqrt{64K^2 + K^2 + 4K^2}
\]

\[
\frac{13}{\sqrt{69}} = \sqrt{69K^2}
\]

\[
\frac{13}{\sqrt{69}} = K\sqrt{69}
\]

\[
D_{69} = \frac{13}{69} = K
\]

\[
N'' = \frac{13}{69}(8\mathbf{i} + \mathbf{j} - 2\mathbf{k})
\]

\[
M = A - N'' = (\frac{69}{69} - \frac{104}{69})\mathbf{i} + (\frac{69}{69} - \frac{13}{69})\mathbf{j} + (\frac{138}{69} + \frac{26}{69})\mathbf{k}
\]

\[
M = -\frac{35}{69}\mathbf{i} + \frac{56}{69}\mathbf{j} - \frac{112}{69}\mathbf{k}
\]
35. Using the values in problem 33, find the vector from A perpendicular to the line BC and terminating on that line.

\[ A = i + j - 2k \]
\[ B = 2i - j + k \]
\[ C = i + 3j - k \]

\[ AB = B - A = i - 2j + 3k \]
\[ BC = C - B = -i + 4j - 2k \]

\[ N = AB \times BC \]
\[ N = -8i - j + 2k \]

\[ |N - h| = |N| |h| \cos 90^\circ \]
\[ |N - h| = 0 \]
\[ |(C - B) \cdot h| = 0 \]

\[ N \cdot h = -8ai - bj + 2ck = 0 \]
\[ (C - B) \cdot h = ai + 4bj - 2ck = 0 \]
\[ -9a + 3b = 0 \]
\[ 3b = 9a \]
\[ b = 3a \]
\[ -8a - 3a + 2c = 0 \]
\[ 2c = 11a \]
\[ c = \frac{11}{2}a \]
35. (Continued)

\[ |h|^2 = a^2 + b^2 + c^2 \]
\[ \frac{69}{21} = a^2 + 9a^2 + \frac{121}{4}a^2 \]
\[ \frac{69}{21} = \frac{161}{4}a^2 \]
\[ \frac{276}{3381} = a^2 \]
\[ \sqrt{\frac{276}{3381}} = a \]
\[ \hat{h} = ai + b\hat{j} + c\hat{k} \]
\[ \hat{h} = ai + 3a\hat{j} + \frac{11}{2}a\hat{k} \]

\[ \hat{h} = \sqrt{\frac{276}{3381}}\hat{i} + 3\sqrt{\frac{276}{3381}}\hat{j} + \frac{11}{2}\sqrt{\frac{276}{3381}}\hat{k} \]

38. Find the value of \( A \cdot B \times C \) if

\[ A = i - j - 6\hat{k}, \]
\[ B = i - 3\hat{j} + 4\hat{k}, \]
\[ C = 2i - 5\hat{j} + 3\hat{k}. \]

\[ B \times C = (-9 + 20)\hat{i} (8 - 3)\hat{j} (-5 + 6)\hat{k} \]
\[ B \times C = 11\hat{i} + 5\hat{j} + \hat{k} \]
\[ A \cdot B \times C = 11 - 5 - 6 \]
\[ A \cdot B \times C = 0 \]
40. If $A, B, C$ are vectors from the origin to the points $A, B, C$ show that

$$AXB + BXC + CXA$$

is perpendicular to the plane $ABC$.

If $AXB + BXC + CXA \perp$ plane $ABC$ then

$$(AXB + BXC + CXA) \times [(A-B) \times (B-C)] = 0$$

$$(AXB + BXC + CXA) \times [AX \times (B-C) - BX \times (B-C)] = 0$$

$$(AXB + BXC + CXA) \times [AXB - AXC - BXB + BXC] = 0$$

$$(AXB + BXC + CXA) \times [AXB + BXC - AXC] = 0$$

$$-AXC = CXA$$

$$(AXB + BXC + CXA) \times (AXB + BXC + CXA) = 0$$

$$0 = 0$$

$\therefore$ $AXB + BXC + CXA \perp$ plane $ABC$.
41. Prove the formula
\[(AXB) \cdot (BXC) \times (CXA) = (ABC)^2\]
\[
(AXB) \cdot (BXC) \times (CXA) = (ABC)^2
\]
\[
(AXB) \cdot [(BCA)C - (BCC)A]
\]
\[
(AXB) \cdot [(B \cdot CXA) \cdot C - (B \cdot CXC ) - A]
\]
\[
(AXB) \cdot (B \cdot CXA) \cdot C
\]
\[
C \cdot (AXB) \cdot B \cdot (CX A)
\]
\[
A \cdot (BXC) = B \cdot (CX A) = C \cdot (AXB)
\]
\[
A \cdot (BXC) \cdot A \cdot (BXC)
\]
\[
(ABC)^2 = (ABC)^2
\]

42. Show that
\[AX(BX C) + BX(CXA) + CX(AXB) = 0.\]
\[
AX(BCX) + BX(CXA) + CX(AXB) = 0
\]
\[
(A \cdot C)B - (A-B)C + (B-A)C - (B \cdot e)A + (C-B)A - (C-A)B = 0
\]
\[0 = 0\]

43. Show that
\[AXB \cdot CXD + BXC \cdot AXD + CXA \cdot B X D = 0.\]
\[
AXB \cdot CXD + BXC \cdot AXD + CXA \cdot BX D = 0
\]
\[
(A \cdot C)(B \cdot D) - (A-D)(B-C) + (B-A)(C-D) - (B \cdot D)(e - A) + (C-B)(A-D) - (C-D)(A-B) = 0
\]
\[0 = 0\]