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Mathematics of Investment

Claudia Morgan Griffin
Ouachita Baptist University

Mathematics of Investment

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Dr. Don Seward

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Claudia Griffin
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Chapter 1

Simple Interest and Simple Discount

Simple Interest:

Final amount = Principle + Interest

$F = P + I$

$I = Prt$, where r is rate and t is time

$F = P(1+rt)$

Simple Discount:

$P = F - I$

$I = Fdt$, where d is discount rate

$P = F(1-dt)$

Simple Discount in Lending Money

If a creditor charges simple discount at the rate d (or interest in advance) on a loan whose maturity payment is F:

1. If F is known, use $I = Fdt$ and $P = F - I$ to find the proceeds P.
2. If P is known, F is obtained by using $P = F(1-dt)$.
3. The rate at which the debtor pays interest is found by use of $I = Prt$.

Exercise 5

3-9 Determine the missing quantity

PROBLEM	TERM	PROCEEDS	MAT. PAYT.	DISCOUNT RATE
3.	3 mos		\$1000	.045
4.	150 days		500	.06
5.	6 mos	\$1250		.05
6.	240 days		1500	.0525
7.	60 days		800	.075
8.	9 mos	1000		.08
9.	8 mos	3000		.04

3. $I = (1000)(.045)(\frac{1}{4}) = 11.25$

$P = 1000 - 11.25 = \$988.75$

4. $I = (500)(.06)(\frac{150}{360}) = 12.50$

$P = 500 - 12.50 = 487.50$

5. $F = \dfrac{1250}{1-(.05)(.5)} = \dfrac{1250}{1-.025}$

$= \dfrac{1250}{.975}$

$F = \$1282$

6. $I = (1500)(.0525)(\frac{240}{360}) = 52.50$

$P = 1500 - 52.50 = 1447.50$

7. $I = (800)(.075)(\frac{2}{12}) = \$10.00$

$P = 800 - 10 = \$790.00$

8. $F = \dfrac{1000}{1-(.08)(\frac{3}{4})} = \dfrac{1000}{1-.06}$

$= \dfrac{1000}{.94} = \$1064$

9. $F = \dfrac{3000}{1-(.04)(\frac{8}{12})}$

$= \dfrac{3000}{1-.027} = \dfrac{3000}{.973}$

$F = \$3082$

18. Prove that $r = \frac{d}{1-dt}$

Let $F = 1$. $I = dt$

$P = 1 - dt$

$$I = Prt$$

$$dt = (1-dt)rt$$

$$rt = \frac{dt}{1-dt}$$

$$r = \frac{d}{1-dt}$$

Discounting Promissory Notes

To discount a promissory note at the discount rate d

1. Compute the maturity value F and the maturity date.

2. Find the term of the discount; let this be t years.

3. Discount F for t years by use of $I = Fdt$, to find $P = F - I$.

Chapter 2
Compound Interest

Compound Interest Formula

$F = P(1+i)^n$, where i is the interest rate per conversion period, P is the original principal, F is the final compound amount to which P accumulates, and n is the number of periods.

$I = F - P$

EXERCISE 7

13. Accumulate \$2000 for 18 years at 3¼% compounded semiannually

Use Table V. Compound Amount of 1.

$i = 1\frac{5}{8} \quad n = 36 \quad (1+i)^n = 1.7866$

$F = 2000(1.7866) = \$3573.20$

Extra-Accumulate \$1500 for 7 years at 4¼ compounded quarterly.

$F = 1500(1+.010625)^{28}$

$\log F = \log 1500 + 28 \log 1.0106$

$= 3.17609 + 28(0.00458)$

$= 3.17609 + 0.12824$

$= 3.30433$

$F = \$2016$

The Discount Problem

$P = F(1+i)^{-n}$

The accumulation factor is $(1+i)^n$.

the discount factor is $(1+i)^{-n}$,

$(1+i)^{-n} = (1-d)^n$

P is called the present value

Exercise 8

7. Discount \$500 for 15 years at 3% compounded monthly.

$500 = F \quad i = (\frac{1}{12})(3\%) = \frac{1}{4}\% \quad n = (15)(12) = 180$

From Table VI, $(1+i)^{-n} = 0.6380$

$P = (500)(0.6380) = \$319.00$

8. Discount \$2500 for $15\frac{1}{2}$ years at $3\frac{1}{2}$ % compounded quarterly. What is the discount?

$2500 = F \quad i = (\frac{1}{4})(\frac{7}{2}) = \frac{7}{8}\% \quad n = (15\frac{1}{2})(4) = 62$

$(1+i)^{-n} = 0.5827$

$P = (2500)(0.5827) = \$1456.75$

19. Discount \$1000 for 15 years

(a) at 5% compounded annually

$P = 1000(0.48101710) = \$481.02$

(b) At 5% simple interest

$$F = P(1+rt)$$

$$1000 = P[1+(.05)(15)]$$

$$P = \frac{1000}{1.75}$$

$$P = \$572$$

(c) At 5% simple discount

$$P = F(1-dt)$$

$$= 1000(1-.75)$$

$$P = \$250$$

Nominal and Effective Rates

j = nominal rate = rate of compound interest quoted
m = number of times compounded per year
w = effective rate = interested earned in one year / principal invested at beginning of year.

To say that x% is the effective interest is eqivalent to saying that principal accumulates at the rate of x% compounded annually.

$$1+w = \left(1+\frac{j}{m}\right)^{m}$$
$$1+w = (1+i)^{m}$$

A statement that money is worth x% means that interest is at the rate x% compounded annually.

Effect of Frequency of Conversion

$1+w = e^{j}$ when interest is converted continually.

Exercise 9

1. Find the effective rate corresponding to the rate 5% compounded

(a) annually $\quad 5\%$

(b) semiannually

$$w = \left(1 + \tfrac{5\%}{2}\right)^2 - 1$$

$$= 0.050625$$

(c) quarterly

$$w = \left(1 + \tfrac{5\%}{4}\right)^4 - 1 = .050945$$

(d) monthly

$$w = \left(1 + \tfrac{5\%}{12}\right)^{12} - 1 = .051162$$

5. What nominal rate compounded semiannually will yield the effective rate 6%?

$$1 + .06 = \left(1 + \tfrac{j}{2}\right)^2$$

$$\sqrt{1.06} = 1 + \tfrac{j}{2}$$

$$\tfrac{j}{2} = 1.029563 - 1 \qquad \text{From Table } \underline{\text{XIV}}$$

$$j = .05913$$

Equivalent rates

When simple interest or simple discount is involved, the sizes of equivalent rates depend on the length of the investment.

Equivalent compound interest rates refer to an investment of a specified principal for one year.

Exercise 10

1. Find the nominal rate which if compounded semiannually is equivalent to 6% compounded quarterly.

$$1+w = \left(1+\frac{.06}{4}\right)^4 = 1.06136355$$

$$\therefore 1.061 = \left(1+\frac{j}{2}\right)^2$$

$$\frac{1}{2}\log 1.061 = \log\left(1+\frac{j}{2}\right)$$

$$\frac{1}{2}(0.02572) = \log\left(1+\frac{j}{2}\right)$$

$$0.01286 = \log\left(1+\frac{j}{2}\right)$$

$$1+\frac{j}{2} = 1.030$$

$$j = .060$$

3. What rate compounded quarterly is equivalent to $(.03, m=2)$?

$$1+w = \left(1+\frac{.03}{2}\right)^2 = 1.0302$$

$$1.0302 = \left(1+\frac{j}{4}\right)^4$$

$$\frac{1}{4}\log 1.0302 = 1+\frac{j}{4}$$

$$\frac{1}{4}(0.0129215) = \log\left(1+\frac{j}{4}\right)$$

$$0.0032304 = \log\left(1+\frac{j}{4}\right)$$

$$1+\frac{j}{4} = 1.0075$$

$$\frac{j}{4} = .0075$$

$$j = .0300$$

12. A man obtains a 2-year loan with simple interest at 6% payable in advance. At what rate compounded semiannually could he just as well pay compound interest?

Sol: $P = \$1$.

$$F = 1+(2)(.06) = \$1.12$$

$$1.12 = 1\left(1+\frac{j}{2}\right)^4$$

$$\log\left(1+\frac{j}{2}\right) = \frac{1}{4}\log 1.12$$

$$= \frac{1}{4}(0.04218) = 0.01054$$

$$1+\frac{j}{2} = 1.0234$$

$$j = .0468$$

Compound interest for n periods, n not an integer

Approximate accumulation of a principal P at compound interest for a term which is not a whole number of interest periods:

1) Find the compound amount on P at the end of the last whole period contained in the given time.

2) Accumulate the rest of Step 1 for the remaining time at simple interest at the given nominal rate.

Approximate discount of an amount F due at the end of a term which is not a whole number of interest periods.

1) Discount F for the smallest number of whole periods containing the given time; this gives the discounted value of F at a date before the actual present.

2) Accumulate the result of Step 1 back to the actual present at simple interest at the given nominal rate.

Exercise 11

1. Find the amount if \$1000 is invested for 2 years and 7 months at 6% compounded quarterly

$F_1 = 1000(1 + 1\tfrac{1}{2}\%)^{10} = 1160.54$

$F_2\ (1160.54)(.06)(\tfrac{1}{12}) = 5.80$

$F = 1160.54 + 5.80$

$= 1166.34$

4. Discount \$2500 for 8 years and 7 months with interest at 6% compounded semiannually.

$P_1 = 2500(1 + 3\%)^{-18} = 2500(.5874)$

$P_1 = 1468.50$

$I = (1468.50)(.06)(\tfrac{5}{12}) = 36.59$

$P = 1468.50 + 36.59 = 1505.09$

Interpolation

An unknown interest rate can be found with sufficient accuracy for most purposes by interpolation in Table V.

Values of Obligations and Their Comparison

Financial obligation - A promise to pay a specified sum of money on a designated date.

The value of a note before the maturity date is found by discounting the maturity value. The value of the note after the maturity date is found by accumulating the maturity value.

Exercise 16

Review for Chapters 1 and 2

4. Find the nominal rate which, if compounded quarterly, yields the effective rate 7%. Do not use logarithms,

$1.07 = (1 + i)^4$

$(1.01675)^4 = 1.06660 \quad i = 0.01723$

$(1.01750)^4 = 1.07186 \quad j = 0.06892$

7. Find the compound interest due at the end of 8 years and 3 months if \$750 is invested at 4%, m = 12

$F = 750\left(1 + \frac{4}{12}\%\right)^{99}$

$F = 750(1.390) = \$1042$

$I = 1042 - 750 = \$292$

13. A debtor owes Ryan the following sums, due without interest: \$1000 due at the end of 2 years and \$2000 due at the end of 5 years. What equivalent single payment would Ryan be willing to accept at the end of 4 years, if money is worth (3%, m = 4) to him?

$1000\left(1 + \frac{3}{4}\%\right)^8 = 1000(1.06160) = \$1,061.60$

$2000\left(1 + \frac{3}{4}\%\right)^{-4} = 2000(0.970554) = 1,941.11$

$\$3,002.71$

15. Find the amount at the end of 9 years and 8 months if $5000 is invested at 5% compounded semiannually, by the approximate method.

$F_1 = 5000\left(1+\frac{5}{2}\%\right)^{19}$

$= 5000(1.5987)$

$F_1 = \$7994$

$I = (7994)\left(\frac{(5)(2)}{(12)(100)}\right) = 67.^{1}$

$F = 8061$

Chapter 3
Annuities With Simple Data

Terminology

Annuity - a sequence of periodic payments which are of equal size unless specified otherwise.

Annuity certain - an annuity whose payments extend over a fixed term of years.

Contingent annuity - an annuity whose payments extend over a period of time whose length cannot be foretold accurately.

Payment interval - the time between successive payment dates. Each payment belongs to the interval which preceeds it.

Term of the annuity - the time from the beginning of the first payment interval to the end of the last one. The first payment is due one payment interval after the beginning of the term.

Annual rent - the sum of the annuity payments in any one year.

Present Value and Amount of an Annuity

The present value of an annuity is the value of the annuity at the beginning of its term.

The amount of an annuity is the value of the annuity at the end of its term.

A = present value

S = amount

$S = A(1+i)^n$

$A = S(1+i)^{-n}$

Geometric Progressions for A and S

If money is worth $(.04, m=2)$, find the present value and the amount of an annuity of \$50 paid at the end of each 6 months for 20 years.

$S = 50 + 50(1.02) + 50(1.02)^2 + \ldots + 50(1.02)^{39}$

$S = 50\left[1 + (1.02) + (1.02)^2 + \ldots + (1.02)^{29}\right]$

Let b = first term $= 1$

Let u = ratio $= 1.02$

Let l = last term $= (1.02)^{39}$

$$S = 50\,\frac{ul-b}{u-1} = 50\,\frac{(1.02)^{40}-1}{.02}$$

$A = 50\left[(1.02)^{-40} + (1.02)^{-39} + \ldots + (1.02)^{-1}\right]$

$b = (1.02)^{-40}$

$u = 1.02$

$l = (1.02)^{-1}$

$$A = 50\,\frac{ul-b}{u-1} = 50\,\frac{1-(1.02)^{-40}}{.02}$$

Annuity With Simple Data

For an annuity problem in which the interest period is the same as the payment interval and the data are as follows:

R is the periodic payment of the annuity
n is the length of the term, expressed in interest periods, or
n is the number of payments of the annuity.
i is the interest rate per conversion period.

The present value A and amount S of the annuity are given by $A = Ra_{\overline{n}|i}$ $\quad S = Rs_{\overline{n}|i}$

$$a_{\overline{n}|i} \equiv \frac{1-(1+i)^{-n}}{i} \qquad s_{\overline{n}|i} = \frac{(1+i)^{n}-1}{i}$$

Exercise 19

1. If money is worth 4%, find the present value and the amount of an annuity of \$1000 payable annually for 25 years.

$i = .04$ Use tables VII and VIII

$n = 25$

$$\text{Amount} = 1000\, s_{\overline{25}|.04} = 1000\,(41.64591) = \$41,645.91$$

$$\text{Present value} = 1000\, a_{\overline{25}|.04} = 1000\,(15.62208) = \$15,622.08$$

10. A purchaser of a farm will pay \$5000 cash and \$500 at the end of each 3 months for 6 years. If money is worth .05 compounded quarterly, find the equivalent cash price of the farm.

$$\text{Price} = 5000 + 500\, a_{\overline{18}|\frac{5}{4}\%} = 5000 + 500\,(16.030)$$

$$= 5000 + 8015$$

$$= \$13015$$

15. A certain investment will yield \$500 at the end of each 3 months for the next 15 years and also 3000 at the end of 15 years. Find the present worth of the project if money is worth 4% compounded quarterly.

$$A = 500\, a_{\overline{60}|1\%} + 3000\,(1 + 1\%)^{-60}$$

$$= 500\,(44.955) + 3000\,(0.5504)$$

$$= 22,477 + 1651 = \$24,128$$

I bought a mobile home for $5900. After subtracting the down payment and adding taxes, license, and the insurance, the principal balance was $5789.50. The total interest was $2282.90, or 10% compounded monthly. This is to be paid in 84 installments of $96.10 each.

Period	Debtor Liability at beginning of period	Interest due	Debtor Liability at end of period
1	5789.50	48.25	5837.75
2	5741.65	47.80	5789.45
3	5693.35	47.40	5740.75
4	5644.65	47.05	5691.70
5	5595.60	46.65	5642.25
6	5546.15	46.20	5592.35
7	5496.25	45.85	5542.10
8	5446.00	45.40	5491.40
9	5395.30	44.90	5440.20
10	5344.10	44.50	5388.60
11	5292.50	44.10	5336.60
12	5240.50	43.70	5284.20
13	5188.10	43.25	5231.35
14	5135.25	42.80	5178.05
15	5081.95	42.40	5124.35
16	5028.25	41.95	5070.20

17	4974.10	41.50	5015.60
18	4919.50	41.00	4960.50
19	4864.40	40.55	4904.95
20	4808.85	40.10	4848.95
21	4752.85	39.60	4792.45
22	4696.35	39.20	4735.55
23	4639.45	38.70	4678.15
24	4582.05	38.25	4620.30
25	4524.20	37.70	4561.90
26	4466.80	37.20	4504.00
27	4407.90	36.75	4444.65
28	4348.55	36.25	4384.80
29	4288.70	35.75	4324.45
30	4228.35	35.25	4263.60
31	4167.50	34.75	4202.25
32	4106.15	34.25	4140.40
33	4044.30	33.75	4077.05
34	3980.95	33.20	4013.15
35	3917.05	32.65	3949.70
36	3853.60	32.10	3885.70
37	3789.60	31.60	3821.20
38	3725.10	31.05	3756.15
39	3660.05	30.50	3690.55
40	3594.45	30.00	3624.45
41	3528.35	29.40	3557.75

42	3461.65	28.85	3490.50
43	3394.40	28.30	3422.70
44	3326.60	27.75	3354.35
45	3258.25	27.20	3285.45
46	3189.35	26.60	3215.95
47	3119.85	26.05	3145.90
48	3049.80	25.45	3075.25
49	2979.15	24.85	3004.00
50	2907.90	24.20	2932.10
51	2836.00	23.65	2859.65
52	2763.55	23.05	2786.60
53	2690.50	22.45	2712.95
54	2616.85	21.80	2638.65
55	2542.55	21.35	2563.90
56	2467.80	20.60	2488.40
57	2392.30	19.95	2412.25
58	2316.15	19.30	2335.45
59	2239.35	18.70	2258.05
60	2161.95	18.00	2179.95
61	2083.85	17.40	2101.25
62	2005.15	16.75	2021.90
63	1925.80	16.05	1941.85
64	1845.75	15.40	1861.15
65	1765.05	14.70	1779.75
66	1683.65	14.05	1697.70

67	1601.60	13.35	1614.95
68	1518.85	12.70	1531.55
69	1435.45	12.00	1447.45
70	1351.35	11.30	1362.65
71	1266.55	10.55	1277.10
72	1181.00	9.85	1190.85
73	1094.75	9.10	1103.85
74	1007.75	8.35	1016.10
75	920.00	7.65	927.65
76	831.55	6.90	838.45
77	742.35	6.15	748.50
78	652.40	5.40	657.80
79	561.70	4.65	566.35
80	470.25	3.90	474.15
81	378.05	3.25	381.30
82	285.20	2.40	287.60
83	191.50	1.60	193.10
84	97.00	.80	97.80

Valuation of a Contract Involving an Annuity

The value of contract is the present value of its future payments. The remaining liability is the present value of unpaid installments.

Exercise 20

1. The buyer of a house will pay \$3500 cash and \$100 at the end of each 3 months for 9 years. If money is worth 5.5% compounded quarterly, find (a) the cash value of the house.

$$P = 3500 + 100\, a_{\overline{36}|1\frac{3}{8}\%}$$

$$= 3500 + 100(28.2451)$$

$$= 3500 + 2824.51$$

$$P = \$6324.51$$

(b) the remaining liability of the debtor just after he pays his 10th installment of \$100.

$$RL = 100\, a_{\overline{26}|1\frac{3}{8}\%} = \$2173.61$$

(c) his remaining liability just before he pays the 10th installment.

$$RL = 100\, a_{\overline{26}|1\frac{3}{8}\%} + 100 = \$2273.61$$

3. In return for a loan, with money worth 5% compounded semiannually, a man promises to pay \$600 at the end of each 6 months for 8 years. (a) Find the sum which he borrows.

$$A = 600\, a_{\overline{16}|.025} = 600(13.05500)$$

$$= \$7,833.00$$

(b) Find his remaining liability just after his 6th payment.

$$A = 600\, a_{\overline{10}|.025} = 600(8.75207) = \$5,251.24$$

Just before his 9th payment.

$$A = 600\, a_{\overline{7}|.025} + 600 = 600(1 + 6.34939)$$

$$= 600(7.34939)$$

$$= \$4409.63$$

4. Suppose that the debtor in Problem 3 failed to make his first 6 payments when they are due. What should he pay to his creditor at the end of 3½ years to cancel his accumulated liability.

$$F = 600\, s_{\overline{7}|.025}$$

$$= 600(7.54743015)$$

$$F = \$4528.46$$

5. Suppose the debtor of problem 3 pays nothing until the end of 3 years. What should he pay then to cancel the debt completely.

$$F = \$7833\,(1+.025)^6 = 7833\,(1.160)$$
$$\doteq \$9,080$$

Solution for an Unknown Periodic Payment

$i + \frac{1}{s_{\overline{n}|i}} = \frac{1}{a_{\overline{n}|i}}$ $\frac{1}{a_{\overline{n}|i}}$ can be looked up in Table IX

$$R\,a_{\overline{n}|i} = A$$
$$R = \frac{A}{a_{\overline{n}|i}}$$

$$R\,s_{\overline{n}|i} = S$$
$$R = \frac{S}{s_{\overline{n}|i}} = S\left(\frac{1}{a_{\overline{n}|i}} - i\right)$$

Exercise 22

7. In order to have a building fund of \$100,000 available at the end of 10 years, a church congregation will invest equal sums at the end of each 3 months at 3½ % compounded quarterly. Find the quarterly investment.

$$R = 100{,}000\left(\frac{1}{a_{\overline{40}|\frac{7}{8}\%}} - .00875\right)$$
$$= 100{,}000\,(0.0297378 - 0.0087500)$$

$$= 100{,}000\,(.0209878)$$
$$R = \$2098.78$$

13. What annuity payable monthly for 7 years can be purchased for 6500 if money is worth 5% compounded monthly?

$$R = 6500 \frac{1}{a_{\overline{84}|\frac{5}{12}\%}}$$

$$= 6500(0.0141339)$$

$$R = 91.87$$

Annuities Due

An annuity due is an annuity whose first payment occurs immediately, on the day to be called present. Each payment belongs to the following payment interval. If the annuity is made in 7 payments,

A = (1st payt.) + (pr. val. of last 6 payts.)

S = (accumulated value of 8 payts.) - (value of the fictitious payt.)

$$A = R + Ra_{\overline{n-1}|i} \qquad S = Rs_{\overline{n+1}|i} - R$$

Deferred Annuities

A deferred annuity is an annuity whose term does not begin until the expiration of a specified time. To say that an annuity is deferred for a certain time means that the term of the annuity begins at the end of this time.

$A = Ra_{\overline{n+h}|i} - Ra_{\overline{h}|i}$, where $\underline{h}$ is the number of deferred payments

$S = Rs_{\overline{n}|i}$

Value of an Annuity on an Arbitrary Date

The value of an annuity on a certain date is the sum of the values of its payments on that date.

For deferred annuities, the present value

$$A = R(a_{\overline{n}|i})(1+i)^{-h}$$

The value of the annuity due

$$\ddot{A} = R(a_{\overline{n}|i})(1+i) \qquad \ddot{S} = R(s_{\overline{n}|i})(1+i)$$

Exercise 27

Chapter Review

1. A man agrees to pay \$100 at the end of each month for 20 years, in purchasing a house. Find the present value of this agreement if money is worth 4% compounded monthly.

$$A = 100\, a_{\overline{240}|\frac{4}{12}\%}$$

$$= 100(165.0218)$$

$$A = 16{,}502.18$$

3. In order to accumulate a fund of \$15,000 by the end of 8 years, what equal deposits should be placed in the fund at the end of each 6 months if interest is at 4% compounded semiannually?

$$R = \frac{15,000}{s_{\overline{16}|2\%}} = 15,000\left(\frac{1}{a_{\overline{16}|2\%}} - .02\right)$$

$$= 15,000(.07365 - .02)$$

$$= 15,000(.05365)$$

$$= \$804.75$$

5. On retirement, a workman finds that his pension calls for payments of \$175 to him (or his estate if he dies) at the beginning of each month for 20 years. Find the present value of this pension at 4% compounded monthly.

$$A = 175a_{\overline{239}|\frac{1}{3}\%} + 175$$

$$= 175(164.5719) + 175$$

$$= 28,800.08 + 175$$

$$= \$28,975.08$$

7. A man now aged 55 is promised a pension of \$500 at the end of each 3 months for 20 years, payable to him, or to his estate if he dies, with the first payment due 3 months after

he is 60 years old. Find the present worth of this promise at 4% compounded quarterly.

$$A = R(a_{\overline{n}|i})(1+i)^{-h}$$
$$= 500(a_{\overline{80}|1\%})(1+.01)^{-20}$$
$$= (500)(54.89)(.820)$$
$$= \$22,500$$

9. Given that $(1.03)^{140} = 62.691904$ and $(1.03)^{-140} = .0159102$, compute $a_{\overline{140}|.03}$ and $s_{\overline{140}|.03}$

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i} \qquad s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

$$a_{\overline{140}|.03} = \frac{1-.0159512}{.03}$$
$$= \frac{.9840488}{.03}$$
$$= 32.80163$$

$$s_{\overline{n}|i} = \frac{62.691904 - 1}{.03} = \frac{61.691904}{.03}$$
$$= 2056.3968$$

11. By use of geometric progressions, derive the usual algebraic expressions for the present value and the amount of an annuity of \$50 payable quarterly for 5 years, if money is worth 6% compounded quarterly.

$$A = 50(1.015)^{-20} + 50(1.015)^{-19} + \dots + 50(1.015)^{-1}$$
$$A = 50\left[(1.015)^{-20} + (1.015)^{-19} + \dots + (1.015)^{-1}\right]$$

b = first term $= (1.015)^{-20}$
u = ratio $= 1.015$
l = last term $= (1.015)^{-1}$

Sum of series $= \dfrac{ul - b}{u - 1}$.

$$A = 50\left[\frac{(1.015)(1.015)^{-1} - (1.015)^{-20}}{1.015 - 1}\right]$$

$$A = 50\left[\frac{1 - (1.015)^{-20}}{.015}\right]$$

$$S = 50 + 50(1.015) + 50(1.015)^2 + \dots + 50(1.015)^{19}$$
$$S = 50\left[1 + 1.015 + (1.015)^2 + \dots + (1.015)^{19}\right]$$

$b = 1$ $\qquad l = (1.015)^{19}$
$u = 1.015$

$$S = 50\,\frac{(1.015)(1.015)^{19} - 1}{1.015 - 1}$$

$$= 50\,\frac{(1.015)^{20} - 1}{.015}$$

13. If money is worth 5% compounded monthly, and if \$2000 is the amount of an annuity whose term is 5 years, find the present value of the annuity.

$$A = S(1+i)^{-n}$$

$$= 2000\left(1+\tfrac{5}{12}\%\right)^{-60}$$

$$= 2000(.779205)$$

$$A = 1558.41$$

15. An endowment fund of \$200,000 is invested at 3% compounded annually, and is designed to pay out \$8000 for college scholarships at the end of each year, until a final date when only a smaller sum will be available. Find the date on which this event will occur.

$$200{,}000 = 8000\,a_{\overline{n}|.03}$$

$$a_{\overline{n}|.03} = 25.000000$$

$$n = 46 \text{ years}$$

Chapter 4
Extinction of Debts by Periodic Payments

Amortization of a Debt

A debt is said to be amortized when all liabilities as to principal and interest are discharged by a sequence of equal payments due at the ends of equal intervals of time.

In current language, the amortization of a debt means the extinction of the debt by any satisfactory set of payments.

The outstanding liability of a debt at any date is the present value of all payments which remains to be made.

To compute the interest and the principal included in any amortization payment:

1) Find the outstanding principal just after the preceding payment.
2) Compute interest for one interval on the principal just obtained in order to find the interest paid on the next date.

3) Subtract the interest found in Step 2 from the periodic amortization payment to find the principal repaid.

EXERCISE 28

1. A debt of $5000 with interest at 6% payable semiannually is to be amortized by equal payments at the end of each 6 months for 3 years. Find the periodic payment and construct an amortization schedule, with entries in the final row accurate to the nearest cent.

Period	Outstanding Prin at Beg. of Period	Interest at 3% Due at End of Period	Total Payment at End of Period	For Repayment of Principal at End of Period
1	5,000.000	150.000	922.988	772.988
2	4,227.012	126.810	922.988	796.178
3	3,430.834	102.925	922.988	820.063
4	2,610.771	78.323	922.988	844.665
5	1,766.106	52.983	922.988	870.005
6	896.101	26.883	922.988	896.105

$$R = 5000 \frac{1}{a_{\overline{6|}.03}} = 5000(.1845975)$$

$$= 922.9875$$

5. A debt of \$2000 will be discharged, interest included, by equal payments at the end of each month for 3 years. If interest is payable monthly at the rate 5%, (a) find the outstanding principal at the beginning of the 3rd year.

$$R = 2000 \frac{1}{a_{\overline{36}|\frac{5}{12}\%}} = 2000(.0299709)$$

$$= \$59.942$$

$$\text{Out Prin} = 59.942\, a_{\overline{12}|\frac{5}{12}\%}$$

$$= 59.942(11.6812)$$

$$= \$700.19$$

(b) find the interest paid on the 30th payment date.

$$\text{Out. prin aft. 29th payt} = 59.942\, a_{\overline{7}|\frac{5}{12}\%}$$

$$= 59.942(6.8848) = 412.689$$

$$(412.689)\left(\frac{.5}{12}\right) = \$1.72 \text{ interest paid}$$

11. A debt of 40,000 will be discharged, interest included by payments of \$2250 at the beginning of each 3 months for 5 years. At what rate is interest payable quarterly?

$$40{,}000 = 2250\, a_{\overline{19}|i} + 2250$$

$$a_{\overline{20}|i} = \frac{37{,}750}{2250} = 16.7777777$$

$$i \approx 1\tfrac{3}{4}$$

Interest is about 5.20% payable quarterly

Amortization, with an Irregular Final Payment

To find the final payment when a debt of \$$A$ is to be discharged, principal and interest included, by equal payments of specified size \$$R$ at the end of each interest period as long as necessary:

1) From $A = Ra_{\overline{n}|i}$ where n is unknown, by mere inspection of Table VIII find when the last regular payment of \$$R$ is due and when the final smaller payment should be made.
2) Find the outstanding principal just after the last regular payment of \$$R$.
3) Add interest for one payment interval to the result of Step 2 in order to determine the final smaller payment.

To find the outstanding principal just after any particular payment, when periodic payments of \$$R$ each are amortizing a debt with original principal \$$A$, write an equation of value on the date of the payment involved, stating that

(outst. princ.) = (accum. val. of \$$A$) - (accum. val. of past payts.)

Exercise 29

1. A debt of $10,000, with interest at 5% payable annually, will be discharged, interest included, by payments of $2000 at the end of each year as long as necessary. Construct the amortization schedule.

Year	Outstanding Prin. at Beginning of Year	Int. Due at End of Year	Total Pay't at End of Year	Prin. Repaid at End of Year
1	10,000.000	500.000	2,000	1,500.000
2	8,500.000	425.000	2,000	1,575.000
3	6,925.000	346.250	2,000	1,653.750
4	5,271.250	263.563	2,000	1,736.437
5	3,534.813	176.741	2,000	1,823.259
6	1,711,554	85.568	1,797.122	1,711.554

3. A debt of $20,000, with interest at 6% payable semiannually, will be discharged, interest included, by payments of $3000 at the end of each 6 months as long as necessary.

(a) Find the outstanding principal at the beginning of the 3rd year.

Let $M be the outstanding principal at the beginning of the 3rd year, or after the 4th payment.

$$20{,}000(1.03)^4 = M + 3000\,s_{\overline{4}|0.3}$$

$$20{,}000(1.12551) = M + (3000)(4.1834)$$

$$22510.2 = M + 12550.8$$

$$M = \$9{,}959$$

(b) Find the final payment.

$3000\, a_{\overline{n}|.03} = 20{,}000$

$a_{\overline{n}|.03} = 6.66666667$

$n = 7$

$M = 20{,}000\,(1.03)^7 - 3000\, s_{\overline{7}|.03}$

$= 20{,}000\,(1.229874) - 3000\,(7.662462)$

$= 24597.48 - 22987.386$

$M = 1610.09$

int due $= (1610.09)(.03) = 48.30$

$H = 1610.09 + 48.30$

$H = 1658.39$ = last payment

Sinking Funds

A sinking fund is one to which payments are made to meet some future need. The amount of the fund at any time is the amount of the annuity formed by the payments. On any payment date, the increase which occurs in the fund is the sum of the new payment and interest on the previous balance in the fund.

Exercise 30

1. A fund is being created by investing \$500 at the end of each 6 months at 4.5% compounded semiannually. How much is in the fund just after the 4th deposit?

$$500 s_{\overline{4}|.0225} = (500)(4.1370364)$$
$$= \$2068.52$$

3. In order to provide \$20,000 for replacement of a machine at the end of 3 years, a corporation will place equal deposits in a fund at the end of each 3 months. Find the periodic deposit if the money is invested at 4.5% compounded quarterly.

$$R\, s_{\overline{12}|1\frac{1}{8}\%} = \$20,000$$
$$R = 20,000(0.08955203 - .01125)$$
$$= 20,000(.07830203)$$
$$R = \$1566.04$$

7. Deposits of \$500 will be invested annually at 3%. How much new interest is added to the fund on the day of the 7th deposit?

Amount just after 4th deposit $= 500s_{\overline{6}|.03} = (500)(6.4684)$

$= \$3234.20$

$(3234.20)(.03) = \$97.03$ = interest on day of 7th deposit

Sinking Fund Method of Retiring a Debt

A creditor may require a loan to be paid in one payment. The debtor may then create a sinking fund to provide the necessary payment. The interest on the principal is paid periodically to the creditor. The principal is paid when due. The book value of a debtor's indebtedness at any time or his net indebtedness is the difference between what he owes and what he has in his sinking fund.

The periodic expense (E) of the debt is

$E = Ai + \frac{A}{s_{\overline{n}|r}}$, where r = the rate at which the sinking fund accumulates

Exercise 31

1. The principal of a debt of \$10,000 will be paid at the end of 8 years by the accumulation of a sinking fund by quarterly deposits, and interest will be payable on the debt quarterly at the rate 6.5%. Find the quarterly expense of the debt to the debtor if his sinking fund is invested at

(a) $(3\%, m=4)$

$$E = (10{,}000)(.01625) + 10{,}000\left(\frac{1}{a_{\overline{32}|\,3/4\%}} - .0075\right)$$

$$E = 162.50 + 10{,}000(.035266 \quad - .0075 \ )$$

$$= 162.50 + 10{,}000(.027766)$$

$$= 162.50 + 277.66$$

$$E = \$440.16$$

(b) $(6.5\%, m=4)$

$$E = 10000\,(.01625) + 10{,}000\left(\frac{1}{a_{\overline{32}|\,.01625}} - .01625\right)$$

$$= 10{,}000\,\frac{1}{a_{\overline{32}|\,.065}} = 10{,}000\,(.040324)$$

$$E = \$403.24$$

(c) $(8\%, m=4)$

$$E = 10{,}000\ (.01625) + 10{,}000\left(\frac{1}{a_{\overline{32}|\,.02}} - .02\right)$$

$$= 162.50 + 10{,}000(.042611 - .02)$$

$$= 162.50 + 10{,}000(.022611) = 162.50 + 226.11$$

$$= \$388.61$$

Approximate Amortization Rate

1) Compute the average interest paid per payment period:

(total interest) $= nR - A$

(ave. int.) $= \dfrac{nR-A}{n}$

2) Assume that the last payment R is wholly principal; then the outstanding principal during the first period is A and during the last period is R:

$$(\text{ave. outst. princ.}) = \frac{A+R}{2}$$

3) An approximation to the interest rate i per payment is:

$$i = \frac{\text{average interest per payment interval}}{\text{average outstanding principal}}$$

The error is approximately $\frac{(n-4)i^2}{6}$, where i is the exact rate.

Exercise 32

3. A bank assists a man in buying an automobile by granting him a \$1000 loan today with the car as security; equal payments are to be made on the loan at the end of each month for 24 months. He is told that the interest charge is 4.5% per year, with an added bookkeeping fee of \$5, and that the sum of the principal and all interest charges divided by 24, is the monthly payment, \$R. Thus, $\$R = \frac{1}{24}(1000 + 90 + 5)$. Find the nominal interest rate at which the payments would pay all principal and interest due at the end of each month on outstanding principal.

$$R = \frac{1}{24}(1095) = \$45.63$$

$$i = \frac{2(nR - A)}{n(A + R)}$$

$$= \frac{2(1095 - 1000)}{24(45.63 + 1000)}$$

$$= \frac{2(95)}{24(1045.63)}$$

$$i = (.0071 \ /\text{month})(12 \text{ months}/\text{yr})$$

$$i = 9.1\%$$

5. A dealer in used cars tells a customer that the cash price of a certain car is \$1025 and that the "time" price is \$1200, payable without added interest by installments of \$75 at the end of each month for 16 months. In spite of absence of mention of an interest rate, at what nominal rate does the buyer pay interest in taking the car at the "time" price?

$$i = \frac{2(nR - A)}{n(A+R)}$$

$$= \frac{2[(16)(75) - 1025]}{16(1025 + 75)}$$

$$= \frac{2(1200 - 1025)}{16(1100)}$$

$$= \frac{(2)(175)}{17,600} = .0199$$

$$\begin{array}{r} 75 \\ 16 \\ \hline 450 \\ 75 \\ \hline \$1200 \end{array}$$

Annual interest = 23.8%

7. A man obtains a \$300 loan, with a term of 18 months, from a bank which specifies an interest charge of 6% per year, meaning that the added charge is 9% (\$300) or \$27, and that the total resulting liability, \$327, will be payable in equal installments at the end of each month

for 18 months. At what nominal rate does the debtor actually pay interest on outstanding principal at the end of each month?

$$i = \frac{2(nR - A)}{n(A+R)} \qquad R = \frac{327}{18} = 18.17$$

$$= \frac{2(327 - 300)}{18(300 + 18.17)}$$

$$= \frac{2(27)}{18(318)} = .00944$$

$i = .944\%$ each month

Exercise 33
Chapter Review

1. A debt is being discharged, interest included at 4.5% payable semiannually, by payments of \$1000 at the end of each 6 months. At the beginning of the 3rd year, the outstanding principal is \$22,050.75. Make up the amortization schedule for the next year.

Interval	Princ.	Int. Due	Total Pay't	Princ repaid
7	\$22,050.75	\$496.14	\$1000.00	503.86
8	21,546.89	484.80	1000.00	515.20

3. A man borrows \$6000 with interest at 5.5% payable annually. He agrees to discharge the debt, interest included, by paying \$750 at the end of each year as long as necessary. When is the last \$750 payment due, and what payment closes the transaction one year later?

$A = R\,a_{\overline{n}|i}$

$6000 = 750\,a_{\overline{n}|.055}$

$a_{\overline{n}|.055} = \frac{6000}{750} = 8.0000$

$n = 10$ yrs. is when last 750 payment is due

$6000(1.055)^{10} - 750\,s_{\overline{10}|.055} = M$

$M = 6000(1.708144) - 750(12.8754)$

$= 10248.86 - 9656.55$

$M = 592.31$

Princ.	Int. Due	Final Payment
\$592.31	\$32.58	\$624.89

5. A house is worth \$18,000 cash. A purchaser will pay \$6000 cash and will discharge the balance, interest included at the rate 5% payable quarterly, by equal payments at the end of each 3 months for 8 years.

(a) Find the quarterly payment

$$R = (18,000 - 6000)\,\frac{1}{a_{\overline{32}|\,1\frac{1}{4}\%}}$$

$$= (12,000)(.038108)$$

$$R = \$457.30$$

(b) What is the unpaid principal just after the 10th quarterly payment?

$$12000(1.0125)^{10} - 457.30\, s_{\overline{10}|\,.0125} = M$$

$$M = 12000(1.1323) - 457.3(10.58\ \ )$$

$$= 13596 - 4830$$

$$M = \$8,760$$

7. At what interest rate payable quarterly will payments of \$225 at the beginning of each 3 months for 6 years amortize a debt of \$4680?

$$\$4680 = 225\, a_{\overline{23}|i} + 225$$

$$a_{\overline{23}|i} = \frac{4455}{225} = 19.8$$

$$i = .0129 \text{ per quarter}$$

$$\text{Interest} = 5.16\% \text{ per year}$$

9. A man buys a farm worth \$25,000 cash. He pays \$3000 cash and agrees to discharge the balance, with interest at 4% compounded semiannually, by a sequence of 20 equal semiannual payments, where the first one is due at the end of 3 years. Find the periodic payment.

$$A = Ra_{\overline{n+h}|i} - Ra_{\overline{h}|i}$$

$$= R(a_{\overline{n+h}|i} - a_{\overline{h}|i})$$

$$R = \frac{A}{a_{\overline{n+h}|i} - a_{\overline{h}|i}}$$

$$= \frac{20,000}{a_{\overline{25}|.002} - a_{\overline{5}|.002}}$$

$$= \frac{20,000}{19.523 - 4.713}$$

$$= \frac{20,000}{14.810}$$

$$R = \$1,350$$

Chapter 5
Bonds

Terminology

A bond is a written contract by a debtor to pay a final redemption value $\$V$ on an indicated redemption date, or maturity date, and to pay a certain sum $\$K$ periodically. The typical bond mentions a borrowed principal, $\$H$, called the face value or par value of the bond, and describes the payments of $\$K$ each as periodic payments of interest on $\$H$ at a specific nominal rate r, to be called the bond rate. In many cases a dated coupon is attached to the bond corresponding to each payment of $\$K$. Such a coupon is a contract to pay $\$K$ on a corresponding date. The coupon annuity is these payments.

A single sum $\$V$ is paid at maturity. An annuity is paid consisting of the periodic coupon payments of $\$K$ each.

Price on an Interest Date

Any date on which a coupon of a bond becomes due is the coupon date or interest date for the bond.

If a bond is sold on a coupon date, the seller takes the coupon payment which is due.

The price \$P the investor (purchaser of a bond) would be willing to pay is the present value of the final redemption payment plus the present value of the remaining coupon annuity payments. This is the general method for computing the value of a bond.

$$P = V(1+i)^{-n} + K\,(a_{\overline{n}|i}$$

H = face value $\qquad$ V = redemp. payt.

K = coupon value $= Hr$ $\qquad$ r = bond rate per period

n = number of periods from purchase date to redemption date.

It is customary to assume that the bond is redeemable at par, and that the interest rate, stipulated by the investor, is compounded as often per year as coupons are payable on the bond.

If $P > V$, the bond is purchased at a premium $P-V$. If $V > P$, the bond is purchased at a discount $V-P$.

Exercise 34

1. A \$1000, 5% bond with annual coupons will be redeemed at the end of 9 years. Find the price to yield

(a) 4%

$$P = 1000(1+.04)^{-9} + (1000)(.05)\,a_{\overline{9}|.04}$$
$$= 1000(0.702587) + 50(7.4353)$$
$$= 702.587 + 371.765$$
$$P = \$1074.35$$

(b) 5%

$$P = 1000(1+.05)^{-9} + 50\,a_{\overline{9}|.05}$$
$$= 1000(.644609) + 50(7.1078)$$
$$= 644.609 + 355.390$$
$$= 999.999$$
$$P = \$1000.00$$

(c) 6%

$$P = 1000(1+.06)^{-9} + 50\,a_{\overline{9}|.06}$$
$$= 1000(.591898) + 50(6.8017)$$
$$= 591.898 + 340.085$$
$$P = \$931.98$$

9. A \$10,000, 4% bond with semiannual coupons is priced to yield 5%. Find the price if the bond is redeemable at par at the end of

(a) 10 years

$$P = 10{,}000(1+.025)^{-20} + (10{,}000)(.02)a_{\overline{20}|.025}$$
$$= 10{,}000(.6102709) + 200(15.58916)$$
$$= 6102.709 + 3117.832$$
$$= 9220.541$$
$$P = \$9220.54$$

(b) 15 years

$$P = 10{,}000\,(1+.025)^{-30} + 200\,a_{\overline{30}|.025}$$
$$= 10{,}000\,(.4767427) + 200(20.93029)$$
$$= 4767.427 + 4186.058$$
$$= 8953.475$$
$$P = \$8953.48$$

(c) 20 years

$$P = 10{,}000\,(1+.025)^{-40} + 200\,a_{\overline{40}|.025}$$
$$= 10{,}000(.3724306) + 200\,(25.10278)$$
$$= 3724.306 + 5020.556$$
$$= 8744.862$$
$$P = \$8744.86$$

Premium or Discount Equation

Premium-discount method for a bond redeemable at par at end of n periods, with face \$H, bond rate r and investment rate i per period.

1. Bond rate greater than investment rate $r > i$. Investor pays

   premium $= (Hr - Hi)\, a_{\overline{n}|i}$

   Price $= H +$ premium

2. $r < i$

   discount $= (Hi - Hr)\, a_{\overline{n}|i}$

   Price $= H -$ discount

In both cases $P = H + (Hr - Hi)\, a_{\overline{n}|i}$

If $V \neq H$, $P = V + (Hr - Vi)\, a_{\overline{n}|i}$

Exercise 35

1. A \$1000, 5% bond with semiannual coupons will be redeemed at the end of 15 years. Find the price to yield the investor (a) 4%

$$P = 1000 + (25 - 20)\, a_{\overline{30}|.02}$$
$$= 1000 + 5(22.3965)$$
$$= 1000.00 + 111.9825$$
$$P = \$1111.98$$

(b) 6%

$$P = 1000 + (25-30)\,a_{\overline{30}|.03}$$
$$= 1000 - 5(19.6004)$$
$$= 1000 - 98.0020$$
$$P = \$902.00$$

5. Dixon borrows \$9000 from Burkhart and signs a note promising to pay interest at the rate 5.5% semiannually and to pay the principal in one installment at the end of 7 years. Burkhart sells the note 5 years before maturity to yield the investor 4%. Find the price paid.

$$P = 9000 + (247.50 - 180.00)\,a_{\overline{10}|.02}$$
$$= 9000 + (67.50)(8.9801)$$
$$= 9000 + 606$$
$$P = \$9606$$

Amortization of a Premium

For a bond bought at a premium, each coupon is greater than the interest due on the investor's principal. The difference between the coupon payment and the interest due is a partial repayment of principal which is available to amortize the premium.

coupon payt. = int. on invest. + amort. payt.

Accumulation of the discount

For a bond bought at a discount, each coupon payment is too small to pay all interest due on the investor's principal. The unpaid interest on each coupon date may be thought of as a new investment in the bond.

int. on princ. = coup. payt. + unpaid int.

int. on princ. = coup. payt. + payt. for accum. of disct.

$\$Q_0$ and $\$Q_1$ represent book values on two successive coupon dates. $\$I$ is simple interest on Q_0 for one coupon period at the investor's rate. $\$K$ is the coupon.

$$Q_1 = Q_0 + I - K$$

Exercise 36

1. A $1000, 5% bond pays coupons on February 1 and August 1 and will be redeemed on 8/1/1965.
a) Find the price of the bond on 2/1/1963 to yield 3%, and form a schedule showing the amortization of the premium.

Date	Int due (1½%)	Coupon	For prem.	Book Value
2/1/63	0.000	0	0.000	1047.826
8/1/63	15.717	25	9.283	1038.543
2/1/64	15.578	25	9.422	1029.121
8/1/64	15.437	25	9.563	1019.558
2/1/65	15.293	25	9.707	1009.851
8/1/65	15.148	25	9.852	1000.00

$$P = 1000 + (25 - 15)\, a_{\overline{5}|.015}$$
$$= 1000 + 10(4.7826)$$
$$= 1000 + 47.826$$
$$= 1047.826$$

b) Without the schedule, find the book value on 2/1/64 to yield 3%.

$$P = 1000 + (25 - 10)\, a_{\overline{3}|.015}$$
$$= 1000 + 10(2.9122)$$
$$P = \$1029.122$$

Professional Practices in Pricing Bonds for Sale

Accrued coupon is the same as accrued interest.
On a day between two successive coupon dates which is ω days after a coupon date,

accrued int. = simp int. for ω days on # at bond rate.

The actual purchase price P on any date is called the flat price.

flat price = accrued interest + quoted price

$$P = kQ$$

When the flat price is described to a buyer by telling him the quoted price, it is said that the bond is being sold at a "price" and accrued interest. Then, the quoted price may be called the "and-interest-price," and is described as a percentage of the par value of the bond. The market quotation is the corresponding rate percent.

When the flat price is specified directly, it is said that the bond is sold flat.

Exercise 37

1. Find the accrued interest on October 16 for a $1000, 4% bond which pays coupons on January 1 and July 1.

$$k = (.04)(1000)(30 \times 3 + 15)\left(\frac{1}{360}\right)$$

$$= \frac{(40)(105)}{360} = \$11.67$$

5. A $100, 6% bond pays interest on April 1 and October 1, and is quoted at 103¼ and accrued interest on November 16. Find the flat price

$$k = (100)(.06)(30+15)\left(\frac{1}{360}\right)$$

$$= \frac{6(45)}{360} = 0.75$$

$$P = 103.25 + 0.75$$

$$P = \$104.00$$

Flat price on a Yield Basis between Interest Dates

To find the flat price and the and-interest-price when a bond is bought to yield a specified rate between interest rates:

1. To obtain the flat price P on the purchase date, find the flat price Q_0 on the preceeding interest date, and add simple interest on Q_0 from then to the purchase date at the investment rate.

$$P = Q_0 + I$$

2. To obtain the quoted price, or and-interest-price Q, subtract from P the accrued coupon to the purchase date.

$$Q = P - k$$

The quoted price of a bond is also called the book value of the bond at the investor's interest rate.

Properties of the Book Value of a Bond

$Q = Q_0 + t(M - K)$

t = part of coupon period

K = periodic coupon payment

M = interest on Q_0 for one coupon period at investment rate i per period.

$P = Q + tK$

Exercise 38

Find the flat price and book value of the bond if it is purchased to yield the specified investment rate. Also, on the date of purchase, find the market quotation of the bond on an "and interest" basis to obtain the specified yield.

Prob	Par value	Bond rate	Interest Dates	Redemp Date	Purch Date	Invest Rate
1.	\$1000	4.2%	6/1; 12/1	6/1/73	8/1/60	5%
3.	5000	3.6%	4/1; 10/1	4/1/66	7/1/61	4.5%
5.	100	6%	1/1; 7/1	7/1/74	10/1/63	6%

(1) $Q_0 = 1000 + (21 - 25)\,a_{\overline{26}|.025}$

$= 1000 - 4(18.9506)$

$Q_0 = 1000 - 75.8024 = 924.198$

$I = \left(\frac{2}{12}\right)(.05)(924.198)$

$= 7.702$

$P = 924.198 + 7.702 = \$931.90$ flat price

$k = (1000)(.042)(\frac{2}{12}) = 7.000$

$Q = 931.90 - 7.000$

$Q = 924.90$ book value

Market quotation $= 92.49\%$

③ $Q_0 = 5000 + 5000(.018 - .0225)a_{\overline{10}|.0225}$

$= 5000 + 5000(-.0045)(8.866)$

$Q_0 = 5000 - 199.485 = 4800.515$

$I = (4800.515)(\frac{3}{12})(.045)$

$= 54.006$

$P = 4800.515 + 54.006$

$= 4854.521$

$P = 4854.52$ flat price

$k = (5000)(\frac{3}{12})(.036)$

$= 45.000$

$Q = 4854.52 - 45.000$

$Q = 4809.52$ book value

$\frac{4809.52}{5000} = 96.19\%$ market quotation

(5) $Q_0 = 100 + 0$

$I = (100)(\frac{3}{12})(.06) = 1.50$

$P = 101.50$ flat price

$k = (100)(.06)(3/12) = 1.50$

$Q = 101.50 - 1.50 = \$100$ book value

100% = market quotation

Approximation to a Bond Yield without Tables

In computing a yield, it is customary to neglect any accrued interest paid at the date of purchase. This part of the flat price is essencially canceled by part of the first interest payment after the purchase date.

Investor's total interest = sum of all coupon payments $\pm$ discount or premium.

To estimate the yield of a bond purchased with a book value Q, n years before the bond is to be redeemed for V,

1. Compute the average invested principal, $\frac{1}{2}(V+Q)$
2. Compute the total interest received by the investor in the n years, and divide by n to find the average interest per year.
3. An estimate j_0 for the yield is

$$j = \frac{\text{average annual interest}}{\text{average investment}}$$

This method gives a result within a few tenths of 1% of the true yield. Express the remaining life of the bond to the nearest month if less than 10 years and to the nearest half-year if over 10 years.

Exercise 39

The quoted price is the book value for the investor. Estimate the yield to tenths of 1%. Assume par value to be $100.

Prob	Redeem at	Quotation	Bond Rate	Life
1	par	120	6%	9 yr
3	par	93	$4\frac{3}{8}$%	15 yr
7	110%	104	3%	9 yr

① $\frac{1}{2}(120 + 100) = 110$ ave invest price

$9(6) - 20 = 54 - 20 = 34$ total int

$\frac{34}{9} = 3.78$ average annual int

$\frac{3.78}{110} = 3.4\%$ yield

③ $\frac{1}{2}(93 + 100) = \frac{193}{2} = 96.5$

$(15)(4\frac{3}{8}) + 7 = 65.63 + 7 = 72.63$

$\frac{72.63}{15} = 4.82$

$\frac{4.82}{93} = 5.2\%$ yield

⑦ $\frac{1}{2}(104 + 110) = 107$

$(9)(3) + 6 = 27 + 6 = 33$

$\frac{33}{9} = 3.67$

$\frac{3.67}{107} = 3.4\%$

Yield by Interpolation

To find the yield i of a bond with a known book value Q,

1. Estimate j by the previous method.
2. Compute the book value at the rate i_1 nearest to j for which the tables can be used. Then, compute the book value for a second rate i_2, chosen so that i probably lies between i_1 and i_2.
3. Determine i by interpolation.

Serial Bond Issues and Annuity Bonds

A serial issue is a bond which is redeemed in installments instead of all on one date. On any date, the flat price of a serial bond issue is the sum of the corresponding prices of all bonds of the issue which remain unredeemed.

An annuity bond, with face value H, is a contract promising the payment of an annuity whose present value is H at the bond rate. At any date the price of the annuity bond is obtained by computing the present value of the future payments of the bond at the investor's interest rate.

Exercise 41

1. A $1,000,000 serial issue of 4% bonds pays coupons on 2/1 and 8/1 and will be redeemed in 5 equal annual installments. The bonds were issued on 2/1/59. An insurance company buys all bonds outstanding on 8/1/61, to yield 5%. Find the price paid.

$$200{,}000\,(.025 - .020) = (200{,}000)(.005)$$
$$= \$1{,}000$$

Bond due in 6 mos $P = 200{,}000 + 1000\,a_{\overline{1}|.025}$

Bond due 2/1/62 $P = 200{,}000 + 1000\,a_{\overline{3}|.025}$

Bond due 2/1/63 $P = 200{,}000 + 1000\,a_{\overline{5}|.025}$

Bond due 2/1/64 $P = 200{,}000 + 1000\,a_{\overline{7}|.025}$

$$P = 800{,}000 + 1000\,(a_{\overline{1}|.025} + a_{\overline{3}|.025} + a_{\overline{5}|.025} + a_{\overline{7}|.025})$$
$$= 800{,}000 + 1000\,(0.975610 + 2.856024 + 4.645828 + 6.349391)$$
$$= 800{,}000 + 14{,}826{,}852$$

Chapter 6
Depreciation, Capitalization, Perpetuities

Depreciation

Depreciation is "that portion of the cost of a machine which is charged off periodically as an expense in operating the business.

Accrued depreciation = sum of depreciation charges to date.

Book value = P - accrued depreciation.

Straight line method of Depreciation

Depreciation charge = ~~change~~ in wearing value × annual rate of depreciation

$$D = dW$$

W = Price - Final salvage value

d = 1/number of years

$$D = \frac{P-L}{n}$$

At the end of t years, the accrued depreciation, E and book value B are

$$E = tD \qquad B = P - tD$$

Sum of the Years' Digits Method

$k = \frac{n(n+1)}{2}$ = sum of the years' digits

Rate of depreciation $= d_K = \frac{D_K}{W}$

The accrued depreciation for the whole life will be exactly W.

Exercise 43

Prepare a depreciation schedule for the asset (a) by the straight line method; by the sum of the years' digits method.

3. A machine costs \$6000 and will have a salvage value of \$500 when retired at the end of 4 years.

(a) $D = \frac{P-L}{n} = \frac{6000-500}{4}$

$= \frac{5500}{4} = 1375$

Year	D	E	B
0	\$ 0	\$ 0	\$ 6000
1	1375	1375	4625
2	1375	2750	3250
3	1375	4125	1875
4	1375	5500	500

(b) $k = 1+2+3+4 = 10$

$d_1 = \frac{4}{10} \quad d_2 = \frac{3}{10} \quad d_3 = \frac{2}{10} \quad d_4 = \frac{1}{10}$

$W = 6000 - 500 = 5500$

$D_1 = \left(\frac{4}{10}\right)(5500) = 2200$

$D_2 = \left(\frac{3}{10}\right)(5500) = 1650$

$D_3 = \left(\frac{2}{10}\right)(5500) = 1100$

$D_4 = \left(\frac{1}{10}\right)(5500) = 550$

Year	D	E	B
0	\$ 0	\$ 0	6000
1	2200	2200	3800
2	1650	3850	2150
3	1100	4950	1050
4	550	5500	500

Depreciation by a Constant Percentage of Declining Book Value

d is an assigned constant annual rate of depreciation.

depr. charge for any yr. = $d \times$ book value at beg of yr.

d is legally $\frac{2}{n}$

n is the life of the machine

$\frac{1}{n}$ is called the natural rate

If the machinery is sold for less than the book value, the extra expense can be claimed. If more is obtained, profit is claimed. The book value at the end of n years

$$L = P(1 - d)^n$$

Exercise 4-1

Prepare a depreciation schedule for the machine by the constant percentage of declining book value method, with the specified annual rate.

1. A machine costs \$6000 and has a life of 6 years. Use twice the natural rate, as permitted by the recent Federal Revenue Act. If the machine is sold as scrap for \$600 at the end of 6 years, what profit or loss on that date in addition to the final depreciation charge will the corporation report?

Year	D	E	B
0	\$ 0	\$ 0	\$6000
1	2000	2000	4000.
2	1333	3333	2667
3	889	4222	1778
4	593	4815	1185
5	395	5210	790
6	263	5473	527

\$73 Profit

Sinking Fund Method of Depreciation

Act as if a fund is to be created by investing equal deposits R at the end of each year for n years, at some interest rate i per year, in order to provide W for replacement.

$$W = R\, s_{\overline{n}|i}$$

depr. charge for any yr. = incr. in depr. fund at end of yr.

$$D_K = R s_{\overline{k}|i} - R s_{\overline{k-1}|i}$$

$R s_{\overline{k}|i}$ = accrued depreciation at end k years.

This method is rarely used in practice

Perpetuity, Simple Case

A perpetuity is an annuity whose payments continue forever.

$$A = \frac{R}{i} = R\, a_{\overline{\infty}|i}$$

Perpetuity with a Payment at end of each K Interest Periods

$$R = \frac{W}{s_{\overline{k}|i}}$$

$$A = \frac{W}{i(s_{\overline{k}|i})}$$

Exercise 46

1. If money is worth 8% compounded quarterly, find the present value of a sequence of payments of \$200 each,

(a) at the end of each 3 months forever.

$$A = \frac{200}{.02} = \$10{,}000$$

(b) at the beginning of each 3 months forever.

$$A = 200 + \frac{200}{.02}$$
$$= \$10{,}200$$

(c) at the end of each 3 months for 25 years

$$A = 200\, a_{\overline{100|}\,.02}$$

$$= 200(43.09835)$$
$$A = \$8619.67$$

5. If money is worth 3%, find the present value of a perpetuity of \$600 per year, if the first payment is due at the end of 5 years.

$$\frac{600}{.03} = 20{,}000$$
$$A = (1+i)^{-4}\, 20{,}000$$
$$= 20{,}000(.88849)$$
$$= 17769.8$$
$$A = \$17{,}770$$

9. It is estimated that maintenance of a certain section of railroad will require \$2000 per mile at the end of each 4 years. If money is worth 3%, find the capitalized cost of this maintenance per mile.

$$A = \frac{2000}{.03(s_{\overline{4}|.03})} = 2000\,(.269027 - .03)\left(\frac{1}{.03}\right)$$

$$= 2000(.239027)\left(\frac{1}{.03}\right)$$

$$= \frac{478.054}{.03}$$

$$A = \$15{,}935$$

13. What income for a university at the end of each year will be provided by an endowment of 200,000 invested at 3% compounded quarterly?

$$C = 200{,}000\,(1 + \tfrac{3}{4}\%)^4$$

$$= 200{,}000(1.03033919)$$

$$C = 206067.838$$

$$W = C - P$$

$$W = \$6067.84$$

Annual Investment Cost

An investor desires interest at the rate i in his major ventures and can reinvest all returned cash at a second rate r, the reinvestment rate. r is usually less than i. He buys an asset at P and plans to sell it at $L < P$ at the end of n years. He desires Pi as annual interest. He foresees a loss of capital $W = P - L$. W is the replacement cost for restoring P at the end of n years. A sinking fund, or capital replacement fund, can be created by investing a replacement deposit R annually at the rate r to supply W at the end of n years. K is the annual cash return which the investor desires from operation of the asset.

K = int on P at rate i + R = annual investment cost

$W = R s_{\overline{n}|r}$

$K = Pi + \frac{W}{s_{\overline{n}|r}}$ at invest. rate i and reinvest rate r

If money is worth i, $r = i$ and $K = Pi + \frac{W}{s_{\overline{n}|i}}$

If $L = 0$, $W = P$, and $r = i$

$$K = P\frac{1}{a_{\overline{n}|i}} \qquad P = K a_{\overline{n}|i}$$

Two different assets are equally economical if their corresponding annual investment costs are the same.

Interest on capt. cost = int on invest + replace deposit to restore capital

$$T = P + \frac{W}{i\, s_{\overline{n}|i}}$$

$$T = \frac{K}{i}$$

Exercise 47

Find the annual investment cost K and also the capitalized cost T of the machine, by first finding K and then using $T = K/i$. Money is worth 4%.

1. Machine with cost \$15,000; life 8 years; final salvage value \$3000.

$$K = (15{,}000)(.04) + \frac{15{,}000 - 3000}{s_{\overline{8}|.04}}$$

$$= 600 + \frac{12{,}000}{9.21} = 600 + 1{,}302$$

$$K = \$1{,}902$$

$$T = \frac{1{,}902}{.04} = \$47{,}550$$

7. A black-top pavement on a street would cost \$10,000 and would last for 5 years with negligible repairs. At the end of each 5 years, \$1000 would be spent to remove the old surface before 10,000 is spent again to lay a new surface. Find the capitalized cost of the pavement, at 5%.

$$K = (10{,}000)(.05) + \frac{10{,}000 - (-1{,}000)}{s_{\overline{5}|.05}}$$

$$= 500 + 11{,}000(0.230975 - .05)$$

$$= 500 + 11{,}000(0.180975)$$

$$= 500 + 1{,}990.73$$

$$K = 2{,}490.73$$

$$T = \frac{2{,}490.73}{.04} = \$49{,}815$$

Valuation of a Wasting Resource

Depletion is the reduction in value of a natural resource.

$$(\text{annual dividend}, K) = \begin{pmatrix}\text{int. at rate}\\ i \text{ on Price, P}\end{pmatrix} + \begin{pmatrix}\text{replace. deposit}, R, \text{ to}\\ \text{return capital}\end{pmatrix}$$

Mine valuation formula

$$P = \frac{K}{i + \frac{1}{s_{\overline{n}|r}}} = \frac{K}{\frac{1}{a_{\overline{n}|r}}} = K a_{\overline{n}|r}$$

final value = \$0

Chapter 7
General Annuity Formulas

The General Problem

The simple case is when the interest period is equal to the payment interval of the annuity. In this case $A = Ra_{\overline{n}|i}$ and $S = Rs_{\overline{n}|i}$

The other two cases are:

I. The payment interval of the annuity may be an integral multiple of the interest period

II. The interest period may be an integral multiple of the payment interval.

Case I

If an ordinary annuity pays \$R at the end of each k interest periods for a term of n interest periods, and i is the interest rate per period, the present value A and and the amount S of the annuity are given by

$$A = \frac{R}{s_{\overline{k}|i}} a_{\overline{n}|i} \qquad S = \frac{R}{s_{\overline{k}|i}} s_{\overline{n}|i}$$

Exercise 49

1. If money is worth 4% compounded monthly, what payment at the end of each month could replace $1000 paid at the end of each year.

$$R\, s_{\overline{12}|\frac{1}{3}\%} = \frac{1000}{s_{\overline{12}|1/3\%}}\, s_{\overline{12}|1/3\%}$$

$$R = 1000\left(\frac{1}{s_{\overline{12}|1/3\%}}\right)$$

$$= 1000(.085150 - .003333)$$

$$= 1000(.081817)$$

$$R = \$81.82$$

7. Payment = $1000
Payment interval = 6 mo.
Term = 15 yr.
Interest rate = 4%, m = 12
Find S.

$$S = \frac{1000}{s_{\overline{6}|1/3\%}}\, s_{\overline{180}|1/3\%}$$

$$= \frac{(1000)(246.1)}{(6.05)}$$

$$S = \$40,700$$

13. The purchaser of a farm will pay \$600 at the end of each 6 months for 7 years. If money is worth 5% compounded monthly, find the equivalent cash price.

$$A = \frac{600}{s_{\overline{6}|\frac{5}{12}\%}} \, a_{\overline{84}|\frac{5}{12}\%}$$

$$= \frac{(600)(70.75)}{6.06}$$

$$A = \$7000$$

General Formulas for A and S

R = periodic payment of the annuity

i = interest rate per conversion period

n = length of term of the annuity in interest periods

$$\rho = \frac{\text{interest period}}{\text{payment interval}}$$

$$k = \frac{1}{\rho}$$

$$A = \frac{R}{s_{\overline{k}|i}}\, a_{\overline{n}|i} \qquad S = \frac{R}{s_{\overline{k}|i}}\, s_{\overline{n}|i}$$

$$\text{and } a_{\overline{k}|i} = \frac{1-(1+r)^{-k}}{r} \qquad s_{\overline{k}|i} = \frac{(1+i)^k - 1}{i}$$

or use Tables XI and XII

Exercise 50

1. An annuity will pay \$200 at the end of each 3 months for 8 years. Find the present value of the annuity if money is worth 4% compounded
(a) semiannually

$$R = 200 \qquad i = .02 \qquad n = 16 \qquad k = \frac{3\text{ mo}}{6\text{ mo}} = \frac{1}{2}$$

$$A = \frac{200}{s_{\overline{\frac{1}{2}}|.02}}\, a_{\overline{16}|.02}$$

$$= (200)(\underline{13.578})(2.0100)$$

$$A = \$5458.10$$

(b) monthly

$R = 200 \quad i = \frac{4}{12}\% = \frac{1}{3}\% \qquad n = 12 \cdot 8 = 96 \qquad k = \frac{3}{1} = 3$

$$A = \frac{200}{s_{\overline{3}|\frac{1}{3}\%}} a_{\overline{96}|\frac{1}{3}\%}$$

$$= \frac{(200)(82.0393)}{3.0100}$$

$A = \$5451.10$

(c) quarterly

$k = 1 \quad R = 200 \quad i = .01 \qquad n = 32$

$$A = 200\, a_{\overline{32}|.01}$$

$$= 200(27.26959)$$

$A = \$5453.92$

9. In place of a payment of \$250 at the end of each month for 4 years, what equivalent payment should be made at the end of 4 years, if money is worth $4\frac{1}{2}\%$ compounded quarterly?

$R = 250 \qquad i = \frac{9}{24}\% = \frac{3}{8}\% \qquad n = 16 \qquad k = \frac{1}{3}$

$$S = \frac{250\, s_{\overline{16}|\frac{3}{8}\%}}{s_{\overline{\frac{1}{3}}|\frac{3}{8}\%}} = (250)(16.4580)(3.0037)$$

$S = \$13,116$

Chapter 8
Life Annuities

A Mortality Table

The statistical probability, or empirical probability, p of an occurrence is the number of times it happened h divided by the number of trials n.

$$p = \frac{h}{n}$$

The probability of the event failing to occur is

$$q = \frac{n-h}{n}$$

$$p + q = 1$$

$_np_x$ = Probability that (x) will live n years
p_x = Probability that (x) will live 1 year
$_nq_x$ = Probability that (x) will die within n years
q_x = Probability that (x) will die within 1 year
x = the age
l = number of people living
d = number of people who died within the allowed time

$$_np_x = \frac{l_{x+n}}{l_x} \qquad p_x = \frac{l_{x+1}}{l_x}$$

$$d_x = l_x - l_{x+1} \qquad _nq_x = \frac{l_x - l_{x-n}}{l_x} \qquad q_x = \frac{d_x}{l_x}$$

Expectation of Life

e_x = curtate expectation of life at age x

$$\mathring{e}_x = \frac{l_{x+1} + l_{x+2} + \dots + l_{99}}{l_x} + \frac{1}{2}$$

Exercise 53

Use Table XV and find the specified probability correct to three decimal places.

1. That a man aged 26 will live at least
   (a) 30 years.

$${}_{30}P_{26} = \frac{l_{56}}{l_{26}} = \frac{740,631}{936,492}$$

$${}_{30}P_{26} = 0.791$$

(b) 50 years.

$${}_{50}P_{26} = \frac{l_{76}}{l_{26}} = \frac{287,973}{936,492} = 0.308$$

3. That a man aged 22 will die within 1 year

$$q_{22} = \frac{d_{22}}{l_{22}} = \frac{2,452}{946,789} = .003$$

5. That a man aged 32 will die in the year after reaching age 50.

$${}_{18}P_{32} = \frac{l_{50}}{l_{32}} = \frac{810,900}{917,880} \qquad q_{50} = \frac{d_{50}}{l_{50}} = \frac{9,990}{810,900}$$

$$q = ({}_{18}P_{32})(q_{50}) = \frac{9,990}{917,880} = .011$$

Present Value of a Pure Endowment

A contingent payment is one whose payment is conditioned on the survival or death of an individual.

An n-year pure endowment is a payment promised to (x) at the end of n years. Its present value of \$1

$$ {}_nE_x = \frac{(1+i)^{-n} \ell_{x+n}}{\ell_x} $$

Of a n-year pure endowment of \$R, the present value

$$ H = R({}_nE_x) = \frac{R(1+i)^{-n} \ell_{x+n}}{\ell_x} $$

$$ D_k = v^k \ell_k \qquad v^k = (1+i)^{-k} $$

$$ H = \frac{R\, D_{x+n}}{D_x} = R\frac{D_y}{D_x} $$

Unless something is said to the contrary, hereafter, it is assumed that $i = 2\frac{1}{2}\%$.

Net single premium for contingent benefits

A life insurance company issues contracts called policies to individuals called policyholders. The company promises to pay certain sums of money, called benefits. If a benefit is contingent on the survival of the policyholder, the present value of the benefit will be found by use of $R({}_nE_x)$. The policy holder makes payments called gross premiums, or simply premiums. The policy date is the day on which the contract is entered into by the company and the policyholder. A set of premiums for a policy is called a set of net premiums if their present value is equal to the present value of the policy benefits, under the assumption that deaths among the policyholders will occur at exactly the rates shown by the mortality table. A net single premium is a single premium payable on the policy date. The gross premiums are the corresponding net premiums plus a certain amount, called loading, which provide for profits and added expense. (The problems will deal with net premiums since each company has its own method of computing gross premiums.)

Exercise 54

3. A man aged 30 is promised a gift of \$10,000 at age 40, if he is alive. Find the present value of this promise.

$$H = \frac{(10{,}000)D_{40}}{D_{30}}$$

$$= \frac{(10{,}000)(328{,}983)}{440{,}800} = \$7463$$

7. A will states that the estate shall be turned over to the heir, now aged 18, when he reaches age 25. If the estate then will be worth \$200,000,

- find the present value of the expectation of the heir, at $2\frac{1}{2}\%$ interest.

$$H = \frac{(200{,}000)D_{25}}{D_{18}}$$

$$= \frac{(200{,}000)(506{,}594)}{612{,}917}$$

$$H = ^{B}165{,}306$$

Life Annuities

A life annuity is one which provides periodic payments on certain dates in case a certain individual(s) remains alive. Ordinary life annuities, or life annuities immediate, are those whose payments occur at the ends of payment intervals after the present date, either for the whole of the life of (x) or for a limited number of years (temporary life annuities). In defferred life annuities and life annuities due, the payments occur at the beginnings of the successive payment intervals.

The net single premium (present value) of a life annuity $L = \frac{N_y}{D_x}$ where $N_K = D_K + D_{K+1} + \ldots + D_{99}$

$$H = R\,\frac{N_y}{D_x}$$

N_K is called a c

Life Annuities Due, and Deferred Life Annuities

$\ddot{a}_x$ = the present value of a whole life annuity due of \$1 paid annually to (x). The first payment of \$1 occurs at age x.

$$\ddot{a}_x = \frac{N_x}{D_x}$$

$$A = R\,\ddot{a}_x = R\frac{N_x}{D_x}$$

A deferred whole life annuity is one in which a sum is paid annually and the 1st payment occurs at the end of n years.

$${}_{n|}\ddot{a}_x = \frac{N_{x+n}}{D_x}$$

$$A = R({}_{n|}\ddot{a}_x) = R\,\frac{N_{x+n}}{D_x}$$

a_x represents the present value of a whole life annuity immediate of \$1 annually to (x). The 1st payment is at age (x+1).

$$a_x = \frac{N_{x+1}}{D_x} \qquad A = R a_x = R\frac{N_{x+1}}{D_x}$$

Exercise 55

Write the net single premium of the life annuity in terms of $\ddot{a}_x$ and $_n|\ddot{a}_x$; finally express the result as a quotient by use of Table XVI.

1. A whole life annuity due of $600 payable annually to a man aged 45.

$$H = R\frac{N_{45}}{D_{45}} = \frac{(600)(5,161,996.00)}{280,638.95}$$

$$H = \frac{5,161,996.00}{467.7316}$$

5. A deferred whole life annuity of $3000 payable annually to a man aged 30 with the first payment at age 60.

$$H = \frac{3000\ N_{60}}{D_{30}} = \frac{3000(1,865,613.58)}{440,800.58}$$

$$H = \frac{1,865,613.58}{146.93353}$$

Temporary Life Annuity

An n-year temporary life annuity due of ₱1 paid annually to a man aged x consists of payments of ₱1 at the beginning of each year for n years to (x), while he remains alive.

$$\ddot{a}_{x:\overline{n}|} = \frac{N_x - N_{x+n}}{D_x}$$

$$A = R\,\ddot{a}_{x:\overline{n}|} = R\,\frac{N_x - N_{x+n}}{D_x}$$

Exercise 56

3. Find the present value of the premiums of a policy in which a man aged 25 will pay ₱100 as a premium at the beginning of each year for 20 years.

$$A = 100\,\frac{N_{25} - N_{45}}{D_{25}}$$

$$= \frac{(100)(12,992,619.10 - 5,161,996.00)}{506,594.02}$$

$$= \frac{7,830623.10}{5065.9402}$$

$$A = ₱1545.74$$

Pension or Annuity Policy

An annuity policy promises a self-created pension to the policyholder.

Benefit - A pension, or deferred whole life annuity, of \$R payable annually, with the first payment at the end of n years.

Premiums - Equal sums of money payable at the beginning of each year for n years, thus forming an n-year temporary life annuity due.

Annual Premium $P = \frac{N_{x+n}}{N_x - N_{x+n}}$

Exercise 57

Do not use the above formulas. All annuities are assumed to consist of equal annual payments. Find the net annual premium, \$P, if it is payable at the beginning of each year as specified, for a life annuity policy with the benefit which is described, for the given age.

3. Man aged 30; the annuity will pay \$2500 at age 65 and annually thereafter for life; premiums payable for 35 years.

$$P\ddot{a}_{30:\overline{35}|} = P\,\frac{N_{30} - N_{65}}{D_{30}} \qquad 2500({}_{20}|\ddot{a}_{45}) = 1000\,\frac{N_{65}}{D_{45}}$$

$$P\,\frac{N_{30} - N_{65}}{D_{30}} = 2500\,\frac{N_{65}}{D_{45}}$$

$$P = \frac{2500\,N_{65}}{N_{30} - N_{65}} = \frac{1000(1,172,129.79)}{10,594,280 - 1,172,130}$$

$$P = \$311.00$$

Chapter 9
Life Insurance

Background for Insurance

An old line or legal reserve company usually engages in the sale of life annuities and pure endowments as well as life insurance. The policy holder is the insured. The benefit is paid to a beneficiary.

Whole Life and Term Insurance

Single premium - one installment paid on the policy date. Assume that any death benefit will be paid at the end of the policy year in which the death occurs. A whole life insurance of \$1 on the life of a man aged x is an agreement by the insurance company to pay \$1 at the end of the year in which he dies.

A_x is the present value or net single ~~income~~ premium of the insurance benefit for a man aged x.

$$A_x = \frac{M_x}{D_x} \qquad M_k = C_k + C_{k+1} + \dots + C_{99}$$

$$C_k = v^{k+1} d_k$$

$$K = RA_x = R\frac{M_x}{D_x}$$

An n-year term insurance of \$1 on the life of (x) promises the payment of \$1 at the end of the year in which (x) dies, only on condition that his death occurs before the end of n years.

$$A^{1}_{x:\overline{n|}} = \frac{M_x - M_{x+n}}{D_x}$$

$$K = R A^{1}_{x:\overline{n|}} = R \frac{M_x - M_{x+n}}{D_x}$$

$A^{1}_{x:\overline{1|}}$ = the natural premium at age x for \$1 insurance.

Exercise 58

1. Find the net single premium for a whole life insurance of \$1000 on the life of a person of age 20.

$$K = 1000 \frac{M_{20}}{D_{20}}$$

$$= \frac{(1000)(196,657)}{580,662}$$

$$K = \$338.68$$

7. How much whole life insurance can a person of age 35 purchase for \$3000 cash?

$$3000 = R \frac{M_{35}}{D_{35}}$$

$$R = \frac{(3000)(381,996)}{174,424}$$

$$R = \$6570$$

Endowment Insurance

An n-year endowment insurance of \$R on the life of a man aged x promises the following benefits:

I An n-year term insurance for \$R; that is, \$R at the end of the year in which the man dies, if his death occurs before the end of n years.

II A pure endowment of \$R payable to the man at the end of n years if he is alive then.

$$K = R A_{x:\overline{n|}} = \frac{R(M_x - M_{x+n} + D_{x+n})}{D_x}$$

Exercise 59

3. Find the net single premium for a \$1000, 10-year endowment insurance for a person aged 35 years.

$$K = \frac{1000(M_{35} - M_{45} + D_{45})}{D_{35}}$$

$$= \frac{1000(174{,}423 - 154{,}736 + 280{,}639)}{381{,}996}$$

$$= \frac{1000(300{,}326)}{381{,}996}$$

$$K = \$786.20$$

Annual Premiums

When premiums are payable annually, they are always payable at the beginnings of the policy years, and cease when the insured person dies.

\$P annually for whole life $\quad H = P\ddot{a}_x = P\frac{N_x}{D_x}$

\$P annually for n years $\quad H = P\ddot{a}_{a:\overline{n}|} = P\frac{N_x - N_{x+n}}{D_x}$

Present value of premiums = Present value of benefits

To find an annual premium, P

1. Write an expression in commutation symbols for the sum of the net single premiums of the policy benefits.
2. Write an expression in commutation symbols for the present value of the premiums.
3. Equate the preceding results and solve for P.

Annual Premium at Age x

Ordinary life policy for \$1 $\quad P_x = \frac{M_x}{N_x}$

n-payment life policy for \$1 $\quad {}_nP_x = \frac{M_x}{N_x - N_{x+n}}$

n-year term policy for \$1 $\quad P^1_{x:\overline{n}|} \; \frac{M_x - M_{x+n}}{N_x - N_{x+n}}$

n-year endowment policy for \$1 $\quad P_{x:\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}}$

Exercise 60

Find the net annual premium for a person of the given age.

3. Man aged 25; a \$2000, 20-payment life policy.

$$_{20}P_{25} = \frac{2000\, M_{25}}{N_{25} - N_{45}} = \frac{(2000)(189,701)}{12,992,619 - 5,161,996}$$

$$= \frac{(2000)(189,701)}{7,830,623}$$

$$= \$48.45$$

7. Man aged 26; a \$1000, 20-year endowment policy.

$$P_{\overline{26}|20} = \frac{M_{26} - M_{46} + D_{46}}{N_{26} - N_{46}}$$

$$= \frac{188,277 + 152,379 + 271,436}{12,486,025 - 4,881,357}$$

$$= \frac{307,334}{7,604,668} = \$40.41$$

Policies of Irregular Type

Present value of premiums = Present value of benefits

Exercise 61

Find the net annual premium of each policy.

1. (a) 10-year term insurance of \$1000 and (b) a pure endowment of \$2000 at the end of 10 years.

Age 27. 10 annual premiums.

Pure endowment $K = 2000 \dfrac{M_{27} - M_{37} + D_{37}}{N_{27} - N_{37}}$

Term insurance $K = 1000 \dfrac{M_{27} - M_{37}}{N_{27} - N_{37}}$

$$\text{Premiums} = \frac{1000\,(3M_{27} - 3M_{37} + 2D_{37})}{N_{27} - N_{37}}$$

$$= \frac{1000[3(186,840) - 3(170,954) + 2(360,161)]}{11,993,210 - 7,757,479}$$

$$= \frac{1000(560,520 - 512,862 + 720,322)}{11,993,210 - 7,757,479}$$

$$= \frac{767,980,000}{4,235,731}$$

$$= \$181$$

Level Premiums and the cost of Insurance

The natural premium for \$R of life insurance at age x ~~as~~ is the net single premium for \$R of 1-year term insurance. (c_x)

$$c_x = \frac{M_x - M_{x+1}}{D_x}$$

Although this increases each year, insurance companies charge a fixed annual premium, or level premium. The over payment is put in a reserve fund. Then money is withdrawn from the fund for underpayment.

Terminal Reserve

The reserve on a policy at the end of any policy year, before the next premium is paid, is called the terminal reserve for that year.

The prospective method for finding a reserve:
net single prem. for remaining benefits = pr. val. of future premiums + terminal reserve.

Exercise 62

1. At an attained age of 50, the net single premium for the remaining benefits of a policy is \$750. There are ten annual premiums of \$50 remaining to be paid, the first due immediately. Find the policy reserve.

$$H. = 750 = 50\,\frac{N_{50} - N_{60}}{D_{50}} + V$$

$$V = 750 - \frac{50(3{,}849{,}488 - 1{,}865{,}613)}{235{,}925}$$

$$= 750 - \frac{(50)(1{,}983{,}875)}{235{,}925}$$

$$= 750 - 420$$

$$V = \$330$$

Chapter 18
Topics from Statistics

The Arithmetic Mean

The arithmetic mean A of k numbers is the sum of the numbers divided by k. A is sometimes called the average. The sum of the differences between the given numbers and A is zero.

Arithmetic mean of a Frequency Distribution

A frequency distribution is $f_1 x_1 + f_2 x_2 + \ldots + f_K x_K$. Sometimes the arithmetic mean of the distribution is called simply the mean value of X and is represented by $\bar{x}$.

Total frequency $N = f_1 + f_2 + f_3 + \ldots + f_K$

$$\bar{x} = \frac{f_1 x_1 + f_2 x_2 + \ldots f_K x_K}{N}$$

The weighted average is computed as above with all f's positive.

Exercise 108

1. Find the arithmetic mean of (3, 5, 7, 18, 24)

$$A = \frac{3+5+7+18+24}{5}$$

$$= \frac{57}{5}$$

$$A = 11.4$$

9. Find the weighted average of (2, 3, 9, 7) with weights (8, 6, 7, 5)

$$\bar{x} = \frac{16+18+63+35}{8+6+7+5}$$

$$= \frac{132}{26}$$

$$\bar{x} = 5.08$$

13. Find the mean weight for the group of people specified in the table.

wt	125	130	135	140	145
People	16	20	31	28	40

$$\bar{x} = \frac{(16)(125)+(20)(130)+(31)(135)+(28)(140)+(40)(145)}{16+20+31+28+40}$$

$$= \frac{2000+2600+4185+3920+5800}{16+20+31+28+40}$$

$$= \frac{18505}{135}$$

$$\bar{x} = 137 \text{ lb}$$

Index Numbers

An index number is a number whose size indicates the relative size, compared to a specific standard, of some other number or group of numbers. Example:

$$\text{index number at date } t = 100 \cdot \frac{X \text{ at time } t}{X \text{ at time } t_0} = R$$

t_0 is the base date.

$$\text{index no. for purchasin power of } \$1 = \frac{100}{X} \cdot 100 = R$$

Exercise 109

1. Refer to the data of Problem 15, Exercise 108. Compute a price index number for the given years for each of the following commodities, with 1950 as the base year: bread; butter; pork chops.

Year	1920	1925	1930	1935	1940	1945	1950	1953	1956
Bread	11.5	9.3	8.6	8.3	8.4	8.8	14.3	16.4	17.8
Index	80.4	65.0	60.1	58.0	58.7	61.5	100.0	114.7	124.5
Butter	70.1	55.2	46.4	36.0	36.0	50.7	72.9	79.0	70.8
Index	96.2	75.7	63.6	49.4	49.4	69.5	100.0	108.4	97.1
Pork Chops	42.3	37.0	36.2	36.1	27.9	37.1	75.4	82.7	67.3
Index	56.1	49.1	48.0	47.9	37.0	49.2	100.0	109.7	89.3

Classified Data

Sometimes given values of a variable X are grouped into classes, each of which extends over a certain numerical range called the class interval. The central value of each class interval is called its class mark. A histogram is a graph of the class intervals vs. frequency.

To find the ~~arithmeth~~ arithmetic mean of a classified distribution:

1. Compute the class mark of each class interval, multiply this mark by the frequency of the class, and add these products.
2. Divide the result of Step 1 by the sum of the frequencies.

Exercise 110

1. The table gives the diameter in inches, at 5 feet above the ground, of the trees measured in a forest survey. Find the average diameter of these trees.

D	23-25	25-27	27-29	29-31	31-33	33-35	35-37	37-39
F	30	62	84	110	130	200	160	125

$$A = \frac{30(24)+62(26)+84(28)+110(30)+130(32)+200(34)+160(36)+125(38)}{30+62+84+110+130+200+160+125}$$

$$= \frac{720+1612+2352+3300+4160+6800+5760+4750}{901}$$

$$= \frac{29454}{901} = 32.7''$$

The Median and the Percentiles

When the N values of the variable X are arranged in descending order, the k^{th} percentile is the smallest number P_K such that $k\%$ of the data are less than or equal to P_K.

To find the k^{th} percentile, where $k < 100$, for N values of X:

1. Arrange the data in order of increasing magnitude. Compute $h = (k\% \text{ of } N)$
2. If h is an interger, tak P_K as the average of the h^{th} and the $(h+1)th$ values.
3. If h is not an interger, let g be the smallest interger larger than h. Then take P_K as the g^{th} value of X.

If x is a value of X, $P_{K-1} < x \leqq P_K$

Exercise 111

1. Find the median and the mean of $(5, 19, 8, 13, 65, 27, 42, 85)$

$$A = \frac{5+19+8+13+65+27+42+85}{8} = \frac{264}{8} = 33$$

$5, 8, 13, 19, 27, 42, 65, 85 \quad h = (50\%)(8) = 4$

$$P_K = \frac{19+27}{2} = 23$$

3. In the distribution of student scores obtained on an examination, the 1^{st}, 85^{th}, 86^{th}, 89^{th}, 90^{th}, + 99^{th} percentiles were, respectively, 15, 103, 107, 115, 119, and 148. Tell the percentile rank of a student whose score was

(a) 12 1

(b) 105 86

(c) 103 85

(d) 118 90

(e) 157 100

Percentiles for a classified distribution

Assume that the values of X are spread out uniformly over the interval. Then use interpolation to find the median.