1970

Mathematics in the Space Program

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MATHEMATICS IN THE SPACE PROGRAM

A Research Paper

Presented to

Miss Kathryn Jones

Ouachita Baptist University

In Partial Fulfillment

of the Requirements for

the Honors Program

by

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May, 1970

Throughout this report reference has been made to Notes On Space Technology compiled by the Flight Research Division, Langley Aeronautical Laboratory, Langley Field, Va. This was loaned to me by a mathematician working at the Manned Space Center, Houston, Texas. It contains math problems used in calculations each day at NASA.

The Notes are arranged under five broad headings. The first four Sections are concerned with Space Mechanics; the next four with Trajectories and Guidance; the next two with Propulsion; the next three with Heating and Materials; and the final four with Space Environment and Related Problems.

Copy of a letter send to: Mr. Burl Kirkland, Landing Analysis Branch, National Aeronautics and Space Administration, Manned Spacecraft Center, Houston, Texas.

Dear Mr. Kirkland,

I got your address from your niece, Marijo Kirkland. She is my roommate at Ouachita. I am Linda Gamble, a junior, majoring in mathematics and Chemistry. I am in the honors program, and I am required to do a special studies paper each semester. Marijo's father told me about your work and some of the mathematical problems you have worked on concerning the space program. The topic of my paper this semester is "Mathematics in the Space Program." Last semester I studied mathematics in computer programing. It was very interesting and tied in with my topic this semester.

I know you are quite busy and any job concerning the space projects now are very involved. If possible, could you send me any information concerning this topic and even some examples of problems that one might be called on to calculate if he worked at the space center.

Any information concerning mathematics would be appreciated. I hope this doesn't inconvenience you any and I'm glad to know someone connected with NASA. It's quite a project and I respect anyone with the mind to accomplish what man has already.

Thank you again.

Sincerely,

Linda Gamble

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A letter received from Mr. Kirkland with a large booklet: NOTES ON SPACE TECHNOLOGY, compiled by the Flight Research Division.

The Space Project has become an almost unbelievable reality of missions that twenty years ago would have been definitely impossible. It demands probably the most technical and precise skills and arts of any other specific task being attempted in the world. Scientists in all fields have combined their efforts to demonstrate human knowledge that has put man in space, orbited him around the earth, let him walk in space, and landed him on the moon to actually walk upon the moon's surface.

Technical knowledge is very advanced and is becoming more so as man conquers so many unknowns. Mathematics is possibly the basis for all calculations that have been involved in these projects that man himself has made successful. All the advanced mathematics, chemistry, etc., began with simple equations, laws of motion, charts, force relations of bodies, biological factors of the earth and even simple algebraic analysis. This beginning has set man in control of mathematical analysis that had never even been mentioned in earlier times. For this reason as math advances, so does the mathematician. Many mathematicians say there is nothing like the mental strain sometimes necessary to keep the space program advancing and successful.

Starting with simple problems pertaining to space, sometimes it is desired to find the motion of a particle with a certain force, F_r, acting on it. The force acting on the particle a long a radius vector is F_r and F_h perpendicular to this vector. The motion of the particle can be obtained by showing the equations $ma_r = F_r$ and $ma_h = F_h$.

These equations are the starting point for studying the orbit of a space craft.

Two bodies of mass may be considered. Assume M is the mass of the earth and m is a satellite. They are attracted by a force proportional to the product of the masses and inversely proportional to the distance between them or $\frac{GmM}{r^2}$.
The force is directed back along r and thus

$F_r = -\frac{GmM}{r^2}$. The differential equations of motion from the two equations $ma_r = F_r$ and $ma_h = F_h$ are:

$$m\left[\frac{d^2r}{dt^2} - r\left(\frac{dh}{dt}\right)^2\right] = -\frac{GmM}{r^2}$$

and

$$\frac{m}{r}\frac{d}{dt}\left(r^2\frac{dh}{dt}\right) = 0 .$$

If there are no forces at right angles to the radius vector then $F_h = 0$.

The above equation can then be integrated:

$$r^2\frac{dh}{dt} = K .$$

If r is moved through an angle dh then the area covered is $dA = \frac{1}{2}r(rdh)$. h here represents an angle θ.[1]

From these and variations of these equations scientists have found that "each planet revolves so that the line joining it to the sun sweeps over equal areas in equal intervals of times"[2]

One of the specific jobs of the mathematician and physicist is the planning of budgets for the different space projects. Budgets, here, doesn't refer only to economical budgets but rather to ascent, descent, or maybe landing budgets. These budgets are initial to any of the actual take-off plans for the space craft.

For example, the Lunar Module operational budgets for the lunar landing mission were published on December 27, 1967. One paragraph from MSC Internal Note No. 68 - FM - 263 from the Manned Spacecraft Center in Houston reads like this;

"The current LM descent and ascent ΔV budgets are 6997 and 6090 fps, respectively, for the lunar landing mission. The 3σ dispersions associated with these budgets are ± 119 fps for descent and ± 12.8 fps for ascent. The primary change is the addition of 12 fps in the braking phase to allow for landing site altitudes below the mean lunar radius. The primary change for ascent is the addition of a contingency bias of 40 fps to cover a PGNCS to AGS switchover situation." [3] ΔV in this paragraph refers to the approximate velocity increment in feet per second.

As in the descent ΔV budget given as 6997 ± 119 fps, a change in this budget affects many calculations. If a budget reflected an increase of 5% then there is a four second deletion in the transition phases and an increase in separation weight of 236 pounds. [4]

Another interesting paragraph considering factors which affect budget planning is the following: "A ΔV allowance of 40 fps has been added to the ascent budget to provide for a PGNCS/AGS switchover. This number was determined by simulation a lunar lift-off with a degraded PGNCS and then switching guidance control to a perfect AGS. Such situation always results in a ΔV penalty since the nominal ascent is designed to be a near-optimim." [5]

Some approximate velocity increments necessary to perform various lunar orbits are indicated below:

	Trajectory	Total ΔV, feet per sec.
1.	Moon Impact	35,000
2.	Circumlunar	35,000
3.	Circumlunar with return to satellite orbit at Earth	47,000

4. Lunar satellite 38.000

5. Landing on Moon 41,000[6]

All of these and many more budgets and factors affecting them and conditions that these budgets affect are all a part of the work carried on by spcae center employees.

Mathematicians in the space program do not follow a year by year schedule. Their work is minute by minute and may call for great mental strain. Take, for instance, the Apollo 13 mission. After having difficulty with their main power source, the men on earth, physicists, mathematicians, chemists were called upon to completely reorganize the return plans for the spaceship.

After trouble was sighted by the crew men of Apollo 13, close analytical work went under way. A mission evaluation team of 150 experts examined photographs and data searching for clues to lead them to make new calculations. It was found that the explosion destroyed the function of the electricity and water producing fuel cells. One of the oxygen tanks exploded when the pressure inside it reached 1,008 pounds per square inch, far below the pressure the tank was designed to contain without rupturing. Normal operating pressure of the tank is around 900 pounds per square inch.[7]

About 105 hours into the flight the technicians on the ground realized the spacecraft was not on a trajectory to get back to earth. They had to make another engine burn to get back on course. The crew was adviced to burn the Lunar Module descent engine to get them back on this trajectory. They were forced to use the lunar lander as a space lifeboat.

Many unique calculations had to be made to get the craft on the right orbit to return to earth. Some of the (ground) people came up with the idea of using the terminator of the earth (the line between sunlight and darkness) to align it.

One of the astronauts himself said, "The cooperation and coordination and the initiative that people have when suddenly faced with an unexpected situation is amazing. They read us up the procedures from the ground. It's amazing that people could respond so fast. This flight has increased my confidence in this nation's space program to take an unusual situation and come out with a successful conclusion." [8]

The space ground crew always has minute to minute accounts of the space flights, what speed they were traveling at a certain time, their distances from the earth and the moon and their orbit of travel. The successful splashdown was hailed as a triumph of American technological skill, courage, and spirit. [9]

From the booklet of guide calculations for different phases of the space program, Section IV has equations that are directed toward the re-entry transfer problem. This is what the "ground" crew was faced with for Apollo 13.

Assuming that we have a vehicle in an orbit entirely outside the main atmosphere, one of the simpliest and more effective ways to cause the vehicle to enter the atmosphere is by firing a retarding rocket at the apogee. The equations which pertain to an apogee kick re-entry are derived as follows:

$$(1)\quad \varepsilon = \left(\frac{V_p^2 r_p^2}{g_0 R^2}\right)\frac{1}{d}$$

$$(2)\quad \frac{K}{d} = V_p - \frac{g_0 R^2}{V_p r_p}$$

$$(3)\quad V_p r_p = V_a r_a$$ [10]

These equations were derived from the following equations:

$$4)\quad \varepsilon = \frac{K^2}{GMd}$$

$$5)\quad V_p = \frac{K}{r_p} = \frac{GM}{K} + \frac{K}{d}$$

$$6)\quad GM = R^2 g_0$$

$$7)\quad K = r_1 V_1 \cos \gamma$$ [11]

E in these equations is a characteristic constant of the curve. It is called the eccentricity. G is the gravitational constant, M is the primary mass, K is Kepler's constant, d represents distance. For a rocket by conditions at a burnout, its distance (r_1) from the origin, its velocity, V_1, and the angle θ, its path makes with the normal to r_1. Rg_0 is the square of a velocity V_0 equal to the velocity for a circular orbit of radius R. A body of mass m has a weight mg_0 at the surface of the earth because of the attractive force of gravity.

These equations may be combined and reduced to give the following combinations:

$$(1) \quad \mathcal{E} = \left(\frac{V_p^2 \, r_p}{g_0 R^2} - 1 \right)$$

$$(2) \quad \mathcal{E} = \left(1 - \frac{V_a^2 \, r_a}{g_0 R^2} \right)$$

$$(3) \quad V_p = V_a \left(\frac{1+\mathcal{E}}{1-\mathcal{E}} \right)$$

$$(4) \quad r_p = r_a \left(\frac{1-\mathcal{E}}{1+\mathcal{E}} \right)$$

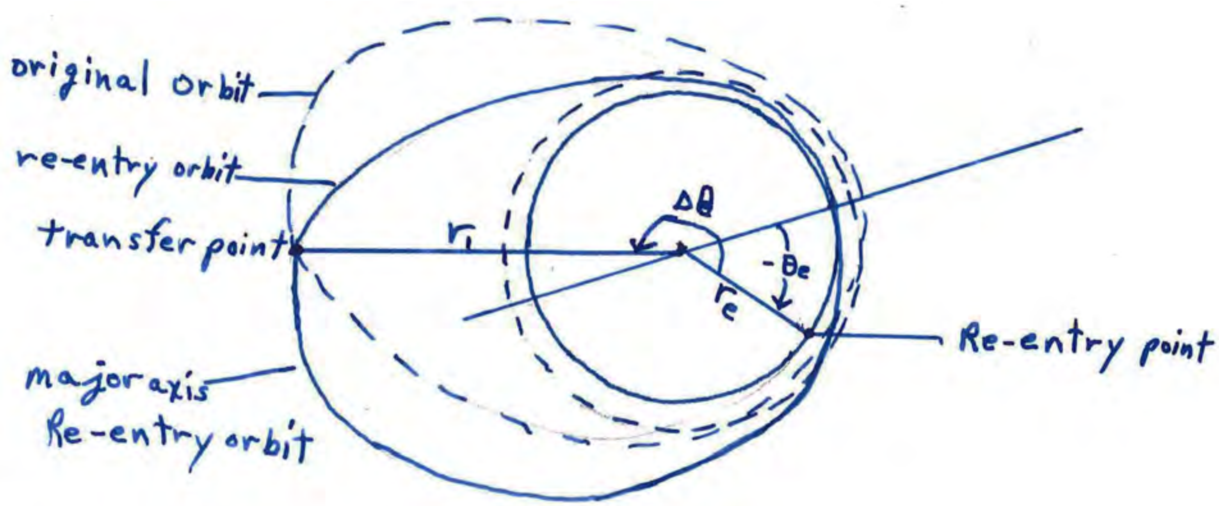
[12]

After the retarding rockets are fired, there is a change in velocity that is effectively instantaneously added. A new semi major axis is established and all are represented by equations. The equation of the new ellipse is:

$$r = \frac{r_{a_1}(1-\varepsilon)}{1+\varepsilon \cos\theta} \qquad 13$$

Now that the new orbit has been established, many more equations must be used to know where the vehicle will enter the atmosphere and what will be the re-entry angle, that is, the angle between the flight path and the horizonal the point of re-entry.

Consider the drawing below:



The velocity impulse required to cause the vehicle to leave its normal orbit and go into the re-entry orbit will be determined. The eccentricity and radius at apogee of the re-entry orbit are given by:

$$(1) \quad \varepsilon = \frac{\tan \gamma_e}{\sin \theta_e - \tan \gamma_e \cos \theta_e}$$

$$(2) \quad r_a = \frac{r_e (1 + \varepsilon \cos \theta_e)}{(1 - \varepsilon)}$$

$$(3) \quad r_a = \frac{r_1 [1 + \varepsilon \cos(\theta_e + \Delta\theta)]}{(1 - \varepsilon)}$$ [14]

Using trignometric identity, substitution, and reduction, the unknown θ_e can be solved. $\theta_e = \tan^{-1}\left(-\frac{A}{B}\right)$ The equation of the re-entry orbit is given by:

$$r = \frac{a(1 - \varepsilon^2)}{1 + \varepsilon \cos \theta}$$ [15]

The velocity and direction of the velocity in the re-entry orbit, the velocity and direction of velocity of the orginal orbit, the velocity impulse for transfer and its direction, and the velocity at re-entry may be obtained by means of these different equations. Problems of this type are calculated for each space flight, the rockets, and satellites that are controlled by our space centers.

There are so many mathematical statements relating to each phase of space maneuvers that it is impossible to mention all of them. Some that are used quite frequently are matrix algebra of transformations, Jacobi's Integral, re-entry equations involving lift, drag, linear inertia, and gravitational forces, Kepler's laws of planetary motion, and Bode's law of distances.

Jacobi's Integral, mentioned above, is the only known integral for the equations of motion. If we have the equation

$$W = \tfrac{1}{2}w^2\ (x^2 + y^2) + GM\ \frac{(1-u)}{r_1} + GM\ \frac{u}{r_2}$$

then the equations of motion can be written as

$$\ddot{x} - 2w\dot{y} = \frac{\partial W}{\partial x}$$

$$\ddot{y} + 2w\dot{x} = \frac{\partial W}{\partial y}$$

$$\ddot{z} = \frac{\partial W}{\partial z}\ .$$

Combining, multiplying, and adding these equations together gives

$$2\ddot{x}\dot{x} + 2\ddot{y}\dot{y} + 2\ddot{z}\dot{z} = 2\dot{x}\ \frac{\partial W}{\partial x} + 2\dot{y}\ \frac{\partial W}{\partial y} + 2\dot{z}\ \frac{\partial W}{\partial z}\ .$$

This equation can be integrated. For example

$$\int \ddot{x}\ \dot{x}\ dt = \int \dot{x}\ d\ \dot{x} = \frac{(\dot{x})^2}{2} \qquad \text{and}$$

$$\dot{x}\ \frac{\partial W}{\partial x}\ dt = \frac{\partial W}{\partial x}\ dx$$

$$dW = \frac{\partial W}{\partial x} dx + \frac{\partial W}{\partial y} dy + \frac{\partial W}{\partial z} dz$$

$$(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2 - 2\int \left\{ \frac{\partial W}{\partial x} dx + \frac{\partial W}{\partial y} dy + \frac{\partial W}{\partial z} dz \right\} + C = 0$$

$$V^2 = 2W - C$$

Substituting the value of W from the first equation given we have

$$V^2 = w^2(x^2 + y^2) + 2GM \frac{(1-u)}{r_1} + 2GM \frac{u}{r_2} - C$$

Jacobi's Integral gives a great deal of information about the motion in an earth-moon system. It is the only known integral for the equations of motion. When the constant of integration has been determined by the initial conditions, the equation for V^2 determines the velocity in the rotating plane at all points in space.[16]

Possibly the most important factor that space engineers are concerned with is that of forces and attractions. Mathematicians have given considerable attention to the point of equal gravitational attraction between the Earth and the Moon. It has been suggested in many sources that a space vehicle need only reach this point of equal attraction to reach the Moon. This equal attraction is represented by the locus of points

$$\frac{r_2}{D} = \frac{-\cos \theta_M + \sqrt{M_1/M_2 - \sin^2\theta_M}}{M_1/M_2 - 1}.$$ [17]

Anyone working with numbers is faced with possible error. Some factors causing error in calculations are negligible but others must be taken into consideration. When scientists began to figure the distance of the moon from the Earth an approximate distance of 238,857 miles was determined. Then the distance was derived by formula considering mean angular velocity, constants, etc. and the distance was found to be 239,074 miles, some 217 miles difference from the first value.

Differences in this meausurement is partly due to the action of the sun on the moon and partly due to neglecting the eccentricity of the moon's orbit. Some factors neglected in the Three Body Problem are:

		ΔV fps
1.	The gravitational field of the Sun	10
2.	Eccentricity of the moon's orbit	45
3.	Inclination of orbit of moon	20
4.	Oblateness of the earth	20
5.	Pressure of Solar radiation.	.04

The ΔV values were found by investigation by Buchheim.

You can see that the effects are small but should be included as corrections to any moon orbit calculations.[18]

Mathematics has been expanded and highly advanced by the progress of the space program.

New mathematical terms, equation relationships, new discoveries in integration and derivatives, and an effort to combine the many fields in math such as trigonometry, algebra, coordinate planes, and solid and analytical geometry are demonstrated in space calculations. The many fields of science are coordinated together to accomplish any aspect dealing with the advanced technical arts of space. Any mathematician can certainly receive a challenge as skills are becoming more technical in science today. It would be a moment of success to a mathematician to know he added some small calculation that placed man on the moon.

FOOTNOTES

[1]Notes on Space Technology, Flight Research Division, Langley Aeronautical Laboratory, Langley Field, Va., 1958, Section 1, p. 3,4, and 5.

[2]Ibid., p. 5.

[3]Kirkland, Burl G., MSC Internal Note No. 68-FM-263, LM Descent and Ascent ΔV Budgets for the Lunar Landing Mission (Apollo Mission G), Landing Analysis Branch, Mission Planning and Analysis Division Manned Spacecraft Center, Houston, Texas, 1968, p. 1.

[4]Ibid., p. 2.

[5]Ibid., p. 5.

[6]Notes, op. cit., Section 3, p. 72.

[7]Shreveport Times, April 21, 1970, Paul Recer.

[8]Shreveport Times, April 22, 1970, Howard Benedict.

[9]Shreveport Times, April 18,1970, Frederick Winship.

[10]Notes, op. cit., Section 4, p. 1.

[11]Ibid., Section 1, p. 2.

[12]Ibid., Section 4, p. 2.

[13]Ibid., Section 4, p. 3.

[14]Ibid., Section 4, p. 15.

[15]Ibid., Section 4, p. 18.

[16]Ibid., Section 3, p. 9, 10, and 11.

[17]Ibid., Section 3, p. 31.

[18]Ibid., Section 3 p. 23, with reference to Buchheim, R. W.; Motion of a small Body in Earth-Moon Space, The RAND Corporation, Research Memorandum RM-1726, June 1956.

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Shreveport Times, April 21, 1970.

Shreveport Times, April 22, 1970