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Radiation Problem

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Honors Project
Math

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Radiation Problem

Given: A sphere of radius 'a' which is radioactive and which has an average range 'b' in the sphere. If, $0<b<2a$, what fraction of total radiation will escape the sphere.

I have decided to divide this into two parts. One, when $b<a$, the other when $b>a$.

The approach to the problem was, that the distance from the centers varied, and this was directly proportional to the radiation that escapes. The distance 'x' was put in terms of 'θ'.

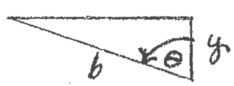

Fig 1.

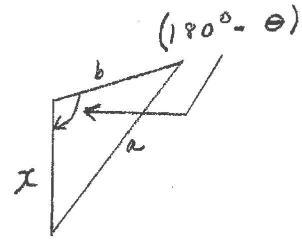

Fig 2

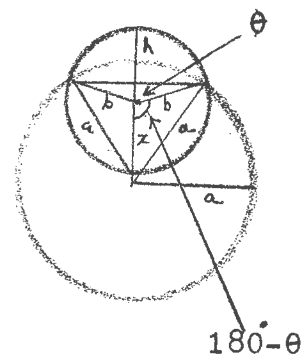

Fig 3

The area outside the large sphere is: $S = 2\pi bh$

The 'h' we have to calculate by the triangle in fig.1.

$h = b-b\cos\theta = b(1-\cos\theta)$

We plug this back into equation 1 and

$S = 2\pi b^2(1-\cos\theta)$

By using the law of cosines we can put the distance 'x' into terms of 'θ'.

$-\cos\theta = \cos(180^\circ-\theta) = (b^2+x^2-a^2)/2bx$

Then:

$1-\cos\theta = (2bx+b^2+x^2-a^2)/2bx$

Now the fraction that escape is: $F=\cancel{2\pi b^2}/\cancel{4\pi}_{2}\, b^2\left(\frac{2bx+b^2+x^2-a^2}{2bx}\right)$

This is then integrated from (a-b) to (a). The total fraction which escapes is:

$$Tf1 = 1/4b \int_{a-b}^{a} (2b + x + (b^2-a^2) / x) \, dx$$

$$= 1/4b \left[2bx + x^2/ 2 + (b^2-a^2) \log x \right]_{a-b}^{a}$$

$$= 1/4b \left[(2ba + a^2/ 2 + (b^2-a^2) \log a) - (2b (a-b)+ (a^2-2ab + b^2) + (b^2-a^2) \log (a-b) \right]$$

$$= 1/4b \left[4ba / 2 + a^2/ 2 - 4ba / 2 + 2ba / 2 + 4b^2/ 2 - a^2/ 2 - b^2/ 2 + (b^2-a^2) \log (a / (a-b)) \right]$$

$$= 1/4b \left[2ba / 2 + 3b^2/ 2 + (b^2-a^2) \log (a / (a-b)) \right.$$

$$= 1/4b \left[ba + 3b^2/ 2 + (b^2-a^2) \log (a / (a-b)) \right]$$

$$= a/4 + 3b/8 + \left[(b^2-a^2)/4b \right] \log (a/(a-b))$$

$$\lim_{b \to 0} Tf1 = a/4 + 3/8(0) + (0-a) \log (a/(a-o))$$

$$\lim_{b \to 0} Tf1 = a/4 + ((0-a)/0) \log(a/(a-o)$$

Let:

$$((o-a/0) \log(a/(a-0) = z$$

$$\text{Lim } z = \text{Lim } f(x)/F(x)$$
$$= \lim_{x \to o} f'(x)/F'(x)$$

Therefore;

$$\text{Lim } z = b/4 \log (a/(a-b)) - a/4 (\log((a/(a-b)/b)$$

$$\lim_{b \to 0} z = (0)\log(a/(a-b)) - a^2/4(\log((a-b)/a)/b)$$

$$= -a^2/4\left(\lim_{b \to 0} (a/(a-b))\right)(1/a)$$

$$= -a^2/4(1/a) = -a/4$$

Therefore:

$$\lim_{b \to 0} z = -a/4$$

And:

$$\lim_{b \to 0} Tf1 = a/4 - a/4 = 0$$

Now I consider the situation when $b > a$:

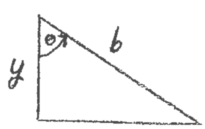

Fig 4.

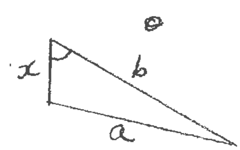

Fig 5

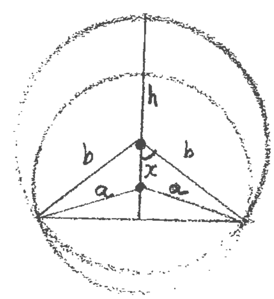

Fig. 6

The radiation outside the sphere with radius 'a' is:

$S = 2\pi bh$

The 'h' we have to calculate by the triangle in fig.4.

$h = b + b\cos\theta = b(1+\cos\theta)$

We plug this into equation 2 and

$S = 2\pi b(1+\cos\theta)$

By using the relationship of the sides and angle of an obtuse triangle, we obtain 'x' in terms of 'θ'.

$\cos\theta = (b^2 + x^2 - a^2)/2bx$

Then:

$$1 + \cos\theta = (2bx + b^2 + x^2 - a^2)/2bx$$

Now the fraction that escapes is :

$$F = S/(4\pi b^2)$$

$$F = \cancel{2\pi} b/\cancel{4\pi} b^2 \, ((2bx + b^2 + x^2 - a^2)/2bx)$$

In the second part we have to consider the instance when the radiation range 'b' is given off from the center. This means there will be a shell of radiation given off outside the radius 'a'. To add this we integrate 1 from 0 to b-a, and add this to our full integral ' Tf2 '.

$$Tf2 = \int_0^{b-a} 1 \, dx + 1/4b \int_{b-a}^{a} (2b + x + (b^2 - a^2)/x) \, dx$$

$$Tf2 = \Big[x \Big]_0^{b-a} + 1/4b \Big[4bx/2 + x^2/2 + (b^2 - a^2) \log x \Big]_{b-a}^{a}$$

$$Tf2 = b - a + 1/4b \left[5ab - 5b^2/2 + (b^2 - a^2) \log (a/(b-a)) \right]$$

$$Tf2 = b - a + 5a/4 - 5b/8 + ((b^2 - a^2)/4b) \log (a/(b-a)$$

Now we set Tf1 = Tf2 taking the limit of both as b approaches a.

$$\lim_{b \to a} Tf1 = \lim_{b \to a} Tf2$$

$$\lim_{b \to a} Tf1 = a/4 + 3a/8 + (a-a)/4a \log (a/(a-a))$$

$$= 2a/8 + 3a/8$$

$$= 5a/8$$

$$\lim_{b \;\; a} Tf2 = a - a + 5a/4 - 5a/8 + (a-a)/4a \log (a/(a-a))$$

$$= 10a/8 - 5a/8$$

$$= 5a/8$$

Now comparing we the results of taking the limits of Tf1 and Tf2:

Since both our answers have an a in them we will divide both of them through by a.

Thus:

$$5/8 = 5/8$$

Our final check will be to tak e the limit of Tf2 letting b approach 2a . This would let all the radiation out, and the equation should approach 1. As before we will have to divide through by a.

$$\lim_{b \;\; 2a} Tf2 = 2a - a + 5a/4 - 5(2a)/8 + (4a-a)/8a \log (a/(2a-a))$$

$$= a + 10a/8 - 10a/8 + 0$$

$$= a$$

Now dividing by a:

$$= 1$$

According to the mean value theorem, when integrating lets say, from 0 to a over a length , you have to divide your answer by 'a'. In a sence this is just what we have done. Therefore, this is our reasoning for doing this operation.

The End