1971

Comparison of Three Schools of Thought in the Foundations of Mathematics

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COMPARISON OF THREE SCHOOLS OF THOUGHT IN THE FOUNDATIONS OF MATHEMATICS

Honors Special Studies
in
Mathematics
for
Dr. D. M. Seward

by
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Spring, 1971

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I.

Some of the most memorable events of the twentieth century took place as a result of conflict. Out of the numerous conflicts staged during this period, only one was resolved not on a common everyday piece of writing paper. The proponents of the conflict--E. V. Huntington, Oswald Veblen, Bertrand Russell, A. N. Whitehead, and David Hilbert--did not use weapons, but they used basic mathematical structure to wage the most extensive and critical investigation into the foundations of mathematics. As a result three schools of thought which are of special prominence and interest were brought to light. These are the postulational school, the logical school, and the formalist school.

The postulational school is led by Professors E. V. Huntington and Oswald Veblen. The specific aim of the school is to establish satisfactory sets of postulates for various branches of mathematics.

The logical school centers around Bertrand Russell and Professor A. N. Whitehead, and their three-volume treatise, Principia Mathematica. The members of this school are interested in the explicit formulation of symbolic logic as a foundation for mathematics.

The formalist school is led by David Hilbert of the University of Gottingen, an eminent mathematician who near the beginning of the century would have been classed as a postulationist. The formalist are attempting to make mathematical proofs rigorous by formalizing the structure of mathematics.[1]

II.

The reasoning underlying the program of the postulational school is simple and amounts to this. Any branch of mathematics must have a starting point somewhere. The postulates as employed, appear in there in perfect light as systems of principles underlying and supporting definite bodies of thought, and so they serve as a model, as an ideal prototype, for the inspiration, the guidance and the criticism of every rational enterprise.[2] Not all of the propositions can be proved and neither can all of its technical terms be defined. In order to completely prove all the propositions, the mathematician must have assumed certain propositions unconsciously and used certain terms glibly without realizing that they were undefined, or else he has been guilty of a "vicious circle" error.

To proceed rationally in the development of a mathematical discipline, it is desirable to make the unproved properties (postulates and theorems) and undefined terms as explicit as possible. Then by logical reasoning, it is possible to define the concepts of the subject in terms of the undefined concepts and deduce further propositions from the unproved propositions. To avoid contradictions it is necessary to adopt a definite restricted set of postulates and a definite restricted

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[1]E. Russell Stabler, "An Interpretation and Comparison of Three Schools of Thought in the Foundations of Mathematics," Mathematics Teacher, 28 (1962).

[2]Cassius J. Keyser, Mathematical Philosophy (New York: E. P. Dutton and Company, 1922).

number of undefined terms.[3]

The methodology is that of generalization by suppression of certain postulates defining a given system. The system defined by the curtailed set of postulates is then developed. At this stage, the undefined terms and the postulates have some concrete or psychological significance to the mind. For example, the postulates may make concrete statements about such undefined terms as points, lines, or numbers.[4] However, if the undefined terms are as they are called, undefined, it must be possible to abstract all previous connotations from them, and to treat them as mere symbols, devoid of any special significance other than what may be implied about them in the statement of postulates. It must also be possible to reinterpret these symbols in new ways. If some new concrete interpretation can be found--which itself appeals to the judgement as being self-consistent--then it is claimed that the postulates are logically consistent.

As an illustration consider the undefined class of elements called "points"; an undefined sub-class of points called "lines"; and an undefined number associated with two points of a line, called "lengths." Assume a knowledge of certain ideas of arithmetic and general language. A point, P, is said to be on a line, L. Two lines are said to intersect if there is a point which belongs to both of them. With these preliminary assumptions, the following postulates are proposed in geometry.

1. Two distinct "lines" intersect in two and only two distinct "points."

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[3]Stabler, op. cit.

[4]E. T. Bell, The Development of Mathematics (New York: McGraw-Hill Book Company, 1945).

2. Through the two intersection "points" of two "lines" pass an infinite number of "lines."
3. The "distance" between two intersecting "points" of two "lines" is the same along all of the "lines" which pass through the two "points."
4. Every "line" has a finite "length" which is equal to the "length" of every other "line."
5. Through two "points" which are not intersection "points" of two "lines" there passes one and only one "line."

These postulates are not altogether easy to comprehend, and a person thoroughly imbued with the traditional view of mathematics would not hesitate to deny their validity, even if reminded that "point," "line," and "length" are undefined terms. However, it will be simpler to remember the abstractness of the original terms used in the postulates, and to reinterpret these words directly. The object is to find some concrete interpretation which will satisfy all five postulates, and it is easy to do this. All that is necessary is to interpret the class of "points" as the class of "points on the surface of a fixed sphere of three dimemsion"; "length" as the concept of "arc length" or "distance" as measured along a great circle of the sphere. The postulates now read as follows:

1. Two distinct spheres intersect in two and only two distinct points of the sphere.
2. Through the two intersection points of two spheres there pass an infinite number of circles.
3. The distance between the two intersection points of two spheres is the same along all of the circles which pass through the two points.
4. Every circle has a finite length which is equal to the length of every other great circle.
5. Through two points of the sphere which are not intersection points of two great circles there passes one and only one circle.

These statements are all quickly judged to be true because of proven theories of Euclidean geometry of the sphere. Furthermore, since Euclidean geometry is self-consistent, it is possible to state that the original five postulates are consistent.

When a tentative list of postulates has been shown to be a consistent set, it is perfectly conceivable that certain postulates of the set are logically deducible from others of the set. Such postulates are superfluous, or redundant. There is no inherent logical fallacy in using a redundant consistent set of postulates, but for at least two reasons it is often desirable that the postulates be free of redundance or independent. First, an independent set of postulates renders the structure of the subject more aesthetically pleasing since no statement is included as a postulate which might be deduced as a theorem. Second, if the redundant postulates are removed, it is possible to go back to any concrete interpretation used in establishing consistency and have fewer postulates to judge true or false than previously. Thus the soundness of the structure of the subject is made to depend more on abstract logical relations, and less on concrete interpretation judgements.

Another characteristic which a consistent set of postulates may or may not possess is that of categoricalness. A set is categorical if it forms the foundation for essentially only one branch of mathematics, while a set is non-categorical if it can serve as a foundation for two or more essentially different branches of mathematics. It would hardly be possible here to give a satisfactory illustration of a categorical set of postulates, but it is not as difficult to cite an example of a non-categorical set.[6]

Start with an undefined class, K, or elements which may be

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[6] Ibid.

designated as A, B, C... Suppose that an undefined operation or relationship between any two elements of the class defines a third element which may or may not belong to the class. The element obtained by operating on A and B is designated as $A \# B$. Now the following postulates are agreed upon:

1. If A and B are elements of the class K, $A \# B$ belongs to K.
2. If A, B, C belong to K, then $(A \# B) \# C = A \# (B \# C)$.
3. There exists a unique element X of K such that $X \# X = X$.
4. For any element A of K there exists a unique element A' of K such that $A \# A' = X$.

Since there are many concrete systems having only a finite number of elements, which satisfy the postulates it is possible to make the following replacements. One permutation, A, replaces x by y and y by x; the other permutation, B, replaces x by itself and y by itself. $A \# B$ will be interpretated as the result of performing permutation A and following it by permutation B.

When interpreting the postulates in accordance with these agreements, the first postulate requires that the result of performing any two permutations of the class successively is a permutation belonging to the class. By trying all the possibilities it is easy to see that this postulate is satisfied. Thus, since A replaces x by y and B replaces y by itself, $A \# B$ replaces x by y and in the same way it replaces y by x; in other words, $A \# B$ is the element A which is known to belong to the class. Similarly, $A \# A = B$, $B \# A = A$, and $B \# B = B$. By the same kind of observations the second postulate is satisfied.

The third postulate requires that a unique "identical" element, X, exists such that $X \# X = X$. The element B meets this requirement inasmuch as $B \# B = B$; furthermore, it is the only element of K which meets the requirement, as A does not.

The fourth postulate requires the existence of unique "inverse" elements, A' and B', for both A and B. Now B is the identical element and is called A # A = B, B # B = B; furthermore, if operations are made on A by B, or on B by A, the identical element is not obtained. Thus, there is a unique inverse for each element of the class--namely, the element itself--and the postulate is satisfied.

This is not judged as a concrete class of two elements, with the accompanying interpretation of the undefined operation, that satisfies all four of our postulates. Since some systems which satisfy the postulates may contain an infinite number of elements, and this system has only two elements, a one-to-one correspondence cannot be set up. Hence the four postulates are non-categorical; they are not sufficient to determine a distinctive mathematical science.

It is not inferred that a set of postulates is not useful if non-categorical. On the contrary there are often advantages in having a non-categorical set. For in this way it is possible to develop parts of a number of separate branches of mathematics at the same time. Thus, in any system which satisfies the postulates for a group, the theorem which can be deduced from these postulates will be true, regardless of whether or not the systems can be put one-to-one correspondence with each other.

The first concern of the school is to establish consistent sets of postulates for various mathematical sciences. It is usually desirable that a set of postulates be independent, and sometimes a set is desired to be categorical, sometimes non-categorical. It is notable that in establishing consistency, independence, and categoricalness, the proofs depend first, on the abstract nature of the postulates when the undefined terms are treated as abstract symbols; second, on the possibility of interpreting the undefined terms, and hence the postulates, in many concrete ways having psychological or intuitive significance; third, on a process of judging that these postulates are satisfied or not satisfied for a given concrete interpretation or system; and fourth, on an assuption that each of the concrete systems used is self-consistent.

To summarize the characteristic features of mathematics from the postulational view point, mathematics is a collection of mathematical sciences whose subject matter may be considered either as abstract, or concrete in innumerable directions. Any mathematical science in completed form is a deductive structure of thought exhibiting a logical chain of reasoning from postulates to theorems, and a corresponding building up process from undefined terms to defined terms. The postulates are not to be considered as self-evident truths, but rather as assumptions concerning fundamental properties which are made in the beginning for the purpose of getting started in the particular branch of mathematics under consideration. It is essential that the postulates be consistent, but absolute proofs of consistency do not seem to be possible. The theorems are not absolutely true, but rather are true at most in relation to the postulates and methods of deductive reasoning used in deriving them.[7]

III.

According to logicalism, mathematics is a branch of logic. Mathematical logic is deductive reasoning as it occurs in mathematics.[8] The

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[7] Ibid.

[8] Bell, op. cit.

starting point is a set of undefined or "primitive ideas," and a group of unproved propositions of logic, whose choice is held to be more or less an arbitrary matter. A preliminary symbolism is adopted for most of the primitive ideas, and most of the primitive propositions are stated in complete symbolic form.

The symbols are at first repellant; they tend to frighten but are not in fact difficult to master. Theoretically, the symbols are not essential but practically they are indispensible as instruments for economizing our intellectual energy.[9] The reduction to symbols is supposed to show the point of application of mathematics, as it were the attachment by means of which it is plugged into its application.[10] It is significant that the primitive ideas and corresponding symbols are not abstract in the sense that the undefined terms or symbols of a branch of mathematics can be considered to be abstract in the postulational school; on the contrary, symbols are used from the beginning to represent concrete logical ideas in concise and convenient form.

Important among the primitive ideas are the following: elementary propositions, elementary propositional functions, assertion, negation, and disjunction. An elementary propostition (designated by p, q, r, etc.) is a statement of the form "this book is green"; an elementary propositional function is a statement with a variable or undetermined element such that when a definite meaning, or value, is assigned to the variable the resulting statement is an elementary proposition. For example, "x

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[9]Cassius J. Keyser, Mathematical Philosophy (New York: E. P. Dutton and Company, 1922).

[10]Ludwig Wittgenstein, Remarks on the Foundation of Mathematics (Oxford: Alden and Mowbrary Ltd., 1967).

is a man" is a propositional function, because if we substitute "this Mr. Brown" or "this dog" the result is an elementary proposition. A property, p, may be asserted to be true (written "$\vdash$ p") or it may be merely considered. The negative of a certain proposition, p, is the proposition "not p" or "$\sim$p." The distinction of two properties p and q if the property "p or " that is "either p is true or q is true," written " $p \vee q$."

The ideas of elementary proposition, negation, and disjunction make the all important definition, the definition implication. The statement "p implies q" written by "$p \supset q$" is defined to mean the same thing as "$\sim p \vee q$" that is "either p is false or q is true," or "if p is true, then q is true."

The notion of implication is prominent in the statement of the primitive propositions. A few of the more significant of these may follow:

1. Anything implied by a true elementary property is true.
2. If, f_1, can be asserted to be true and we can assert that $f_1 \supset f_2$, then we can assert f_2 is true.
3. $(p \vee p) \supset p$ (p or p implies p)
4. $q \supset (p \vee q)$ (q implies p or q)
5. $(p \vee q) \supset (q \vee p)$

The primitive properties should be referred to as assertions of primitive propositional functions, for the letters p, q stand for variable or undetermined elementary propostions, and it is asserted that the statements hold for every specific property which may be substituted for p, q.

Some of the significent theorems in the theory of deduction which are deduced from the primitive propositions are the following:

1. $(p \supset \sim p) \supset \sim p$ "if p implies not p, then p if false"
2. $[p \supset r)] \supset [(p \supset q) \supset (p \supset r)]$ "if q implies r, then if p implies q, p implies r.
3. $p \vee \sim p$ "p is true or p is false"

4. $p \supset \sim(\sim p)$ "p implies that not p is false"
5. $(p \supset q) \supset (\sim q \supset \sim p)$ "if p implies q, then not q implies not p"

All of these theorems are seen to correspond to methods of deductive reasoning which are usually taken for granted. There is not quarrel with the postulational view that the method and structure of mathematics are deductive, but in the postulational school the nature of deductive reasoning remains largely unanalyzed, while in the logical school the deductive methods and concepts are themselves developed in great detail from a foundation of undefined terms and unproved properties of logic. Furthermore, instead of viewing the subject matter of mathematics as wholly abstract, the logical school looks upon mathematical subject matter as consisting of any concepts which may be ultimately traced back and defined in terms of the undefined concepts of logic. Mathematics is not now a collection of deductive sciences, each with its own foundation, but a single unified deductive science with a single foundation in logic.

Symbolic logic has established the thesis that all existing mathematics (and presumable all potential mathematics) is literally a logical outgrowth of a few primitive ideas, and a few primitive propositions of logic.[11] Unrestricted in subject matter, logic analyzes its propositions as referring to classes and attributes.[12] No relationship can be defined without a logical frame and any apparent disharmony in the description of experiences can be eliminated only by an appropriate widening of the

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[11]Keyser, op. cit.

[12]Arthur Pap, "Mathematics, Abstract Entities, and Modern Semantics," Scientific Monthly, 85 (1957), 32.

conceptual framework.[13]

IV.

Finally, the formalist school, like the logical school, is attempting to carry the ultimate foundation of mathematical knowledge further back than the postulational school. At the same time, they are trying to establish the consistency of all mathematics, and thus are attacking a problem which is not explicitly investigated by the logical school. Hilbert contends that the ultimate foundation for mathematics lies not in logic, but in certain pre-logical objects which are preliminary conditions for logical thinking, and about which seem to be viewed with a definite intuitive knowledge.

Certain mathematical statements, made by the use of symbols, are immediately capable of verification by the intuitive method, because of the inherent nature of the concepts represented by the symbols $3 + 1 = 1 + 3$. This statement is an example of a real proposition.

On the other hand, certain other mathematical statements, like $a + 1 = 1 + a$ where a represents any integer, are not verified in this way, because it is impossible to test all possible integers in the equation. To avoid the difficulty this equation must be thought of as a purely formal statement, and if it is to be verified at all it must be verified by formal argument without regard to the meaning of the statement. Statements of this second type are examples of ideal propositions, and the formal agruments necessary to establish them are to be made, according to definite rules, from previously listed axioms, with

[13]Niels Bohr, "Mathematics and Natural Philosophy," Scientific Monthly, 82 (1956), 86.

reliance now merely on our intuitive knowledge of the characteristics of marks as marks. Examples of axioms are:

1. $A \rightarrow (B \rightarrow A)$
2. $B \rightarrow A \vee B$
3. $A \rightarrow A$
4. $A = a$
5. $a = b \rightarrow A(a) \rightarrow A(b)$
6. $a = o$

As the specific foundation for this formalized structure the formalist propose axioms which are the images of fundamental logical and mathematical ideas, concerned with implication and ordinary integers. By following formal rules which are so chosen as to correspond with accepted processes of deductive thinking theorems are deduced from the axioms. These again are images of corresponding theorems having thought content. New axioms are introduced as a basis for continuing the process, provided at each state the consistency of the axioms is established. The method of proving consistency is also a special formula procedure based on two of the original axioms. In other words, every proof should be so reconstructed as to make apparent a particular kind of formal structure which can be discovered by appropriate restatement of it.[14]

The purpose of the formalized procedure is not to make mathematics an arbitrary game with meaningless marks, but to render the logical structure of existing mathematics more secure by making it more definitely objective. This theory makes explicit the rules according to which thinking proceeds, and thereby provides a basis for objective thinking in all fields, as opposed to subjective opinion and emotion.

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[14]Constantine Plitis, "Limitations of Formalization," <u>Philosophy Science</u>, 32 (1965).

V.

First, a general survey from the postulational standpoint, mathematics is a collection of deductive sciences each having its own set of postulates and undefined terms, each making free use of logic in developing its own set of theorems; from the logical standpoint, mathematics is a unified science which can be developed out of logical concepts from logical principles, and by use of logical principles; from the formalist standpoint, the formal structure of mathematics is to be developed from certain logical and mathematical axioms, considered as images of thought, by means of formal application of the rules of deduction.

From all three standpoints, the method and structure of mathematics may be called deductive, for, in each case, the program calls for assumptions and undefined terms as a starting point for the use of deductive reasoning to arrive at new conclusions. In the postulational school there are different starting points for the various mathematical sciences, most of which assume deductive logic without analyzing it; in the logical school the starting point is carried down into the primitive ideas and propositions of logic, and logic is then developed in great detail, finally merging with mathematics; in the formalist school, the most fundamental level of the foundations goes still deeper, and consists in our intuitive knowledge of pre-logical and pre-math symbols, while the next higher level consists of axioms both of mathematics and logic.

It seems fair to say that any judgement concerning the truth or consistency of the assumptions used in any of the three schools, depends in the last analysis, on intuitive, unproved notions. Any absolute basis for claiming truth or consistency thus seems to be lacking.

Therefore, certain general conclusions concerning the nature of mathematics, when viewed in the light of modern investigations, can be drawn.

1. The subject matter of mathematics is not restricted to ideas of "number and space." From the modern point of veiw the subject matter may include logic, abstract sciences, and a wide range of concrete interpretations. The ultimate origin of the subject matter seems to be in intuitive ideas.
2. A starting point consisting of assumptions and undefined terms is necessary for the development of any mathematical structure. No matter whether this starting point is explicitely formulated in logic, beyond logic, or prior to logic, any judgement concerning the truth of the assumptions seems to depend on intuitive considerations.
3. The method by which a mathematical structure is developed is the method of deductive reasoning used in obtaining theorems from the fundamental assumptions, and the corresponding process of defining new concepts with ultimate dependence on the original undefined concepts.
4. The soundness of a mathematical structure of thought depends on the soundness of the deductive reasoning used in developing it, and on the consistency of the original assumptions. So far no absolute basis has been established for judging whether these requirements are met.
5. The theorems of mathematics are not absolute truths. They are true at most in relation to the postulates from which they were deduced, and the methods of reasoning by which they were deduced.

VI.

The contrast of these conclusions with the traditional view of mathematics is striking but it is not safe to claim that a presentation of any final picture of the nature of mathematics has been made; for the fundamental concepts and methods of mathematics are perpetually in a state of evolution and conflict.[15]

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[15] Stabler, _op. cit_.

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