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# An Introduction to Linear Programming

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# LINEAR PROGRAMMING

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# INTRODUCTION

This paper represents a study of the text An Introduction To Matrices, Vectors, and Linear Programming. It is composed chapter by chapter taking the more important statements, definitions, and theorems from each and then working out exercises to illustrate their meaning. Other exercises were worked in the course of the study than are included in this paper but these were selected as brief illustrations of the type of problems that were worked.

CHAPTER ONE. INTRODUCTION TO MATRICES.

There are two kinds of mathematical elements: matrix and vector. A matrix A is a rectangular array of elements, denoted by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix}$$

An nxn matrix is said to be of order n. Two matrices A and B are said to be equal when they are of the same order and all their corresponding entries are equal. Likewise, a real matrix A is greater than a real matrix B of same order when each entry of A is greater than the corresponding entry of B. Matrices can be added only when they are the same order or conformable for addition. To multiply a matrix by a scalar multiply each element of the matrix by the scalar. When multiplying two matrices they must be conformable for multiplication or the number of columns of A must equal the number of rows of B. For example, if A is a 1 x p matrix and B is a p x 1 matrix they are conformable for multiplication and the product C = AB is a 1 x p matrix. Each entry $C_{ij}$ of C is obtained by multiplying corresponding entries of the ith row of A and the jth column of B and then adding the results.

EXERCISES:

1. Find, if possible, all values for each unknown that will make each of the following true:

(a) $\begin{bmatrix} 2 & 4 \\ 5 & x \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 7 \end{bmatrix}$ $x = 7$

(b) $\begin{bmatrix} 2 & 3 \\ x & -1 \end{bmatrix} > \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}$

$x > 8$

(c) $\begin{bmatrix} 2 & 3i \\ x & -1 \end{bmatrix} > \begin{bmatrix} 0 & i \\ 8 & -2 \end{bmatrix}$ not possible.

2. Perform the addition. $\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & -3 \end{bmatrix}$

3. Calculate, if possible, the following.

$$\begin{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} & \begin{bmatrix} -1 & 3 \end{bmatrix} \\ \begin{bmatrix} 5 \\ 7 \end{bmatrix} & \begin{bmatrix} -5 \\ 0 \end{bmatrix} \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} 0 & -3 \end{bmatrix} & \begin{bmatrix} 3/4 & 8 \end{bmatrix} \\ \begin{bmatrix} 66 \\ -2 \end{bmatrix} & \begin{bmatrix} -1 \\ -2 \end{bmatrix} \end{bmatrix} = \begin{matrix} \begin{bmatrix} 1 & -1 \end{bmatrix} & \begin{bmatrix} -\frac{1}{4} & 11 \end{bmatrix} \\ \begin{bmatrix} 71 \\ 5 \end{bmatrix} & \begin{bmatrix} -6 \\ -2 \end{bmatrix} \end{matrix}$$

4. Given $A = \begin{bmatrix} 2 & -1 \\ -3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 \\ -1 & 3 \end{bmatrix}$ Calculate:

(a) $3A = \begin{bmatrix} 6 & -3 \\ -9 & -12 \end{bmatrix}$ (b) $A + 3B = \begin{bmatrix} -4 & -1 \\ -6 & 8 \end{bmatrix}$

5. Multiply: $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 8 & -5 \end{bmatrix}$

6. Multiply: $\begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 2 \end{bmatrix}$

## CHAPTER TWO. INTRODUCTION TO VECTORS.

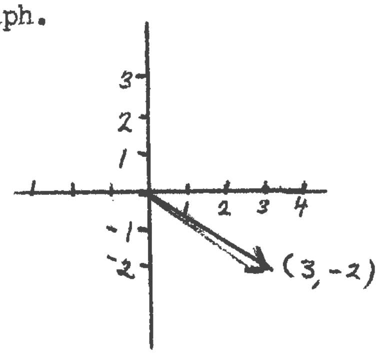
A vector $\alpha$ of order n is an ordered set of n scalars, $( a_1, a_2, \ldots a_n )$. The $a_i$'s are components of $\alpha$, and for n components we say $\alpha$ is an n-dimensional vector. The addition of two vectors is called their resultant. The length of the line segment is the magnitude of $\alpha$ designated $|\alpha|$ and $|\alpha| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Direction is indicated by an arrow and expressed by cosines of direction angles: $\cos\theta = \frac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$

$$\cos\theta_2 = \frac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \qquad \cos\theta_3 = \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

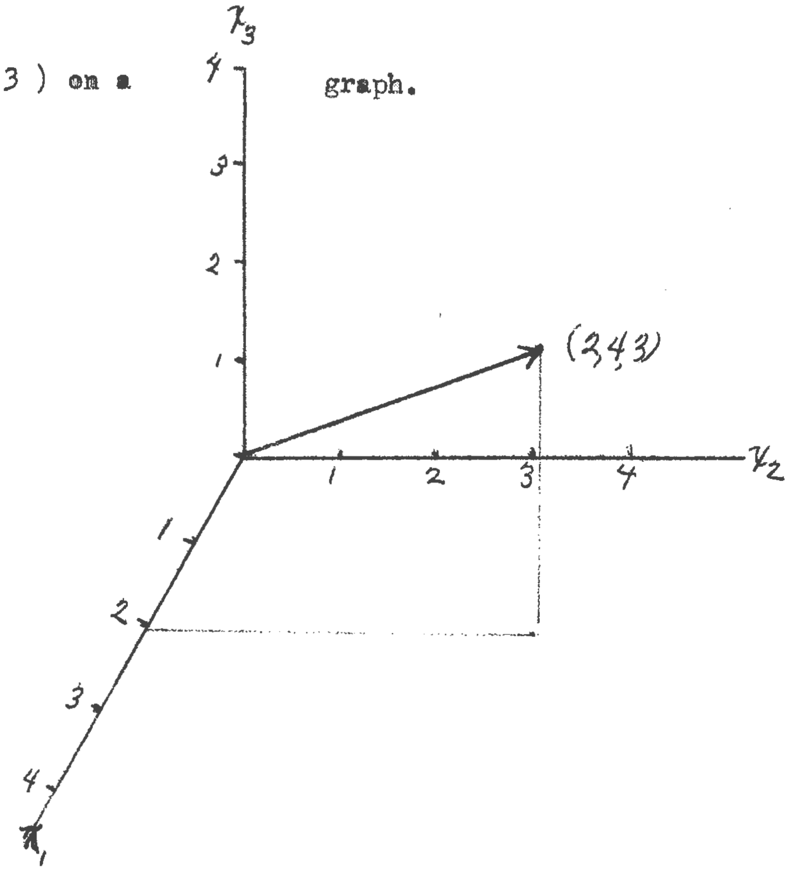
An important set of unit vectors in three dimensional space is $i = ( 1,0,0 )$ $j = ( 0,1,0 )$ $k = ( 0,0,1 )$. Multiplying a vector by a negative scalar changes the direction. The scalar product of two vectors is derived by multiplying corresponding components and then adding these products. If $\alpha$ and $\beta$ are two nonzero vectors in the $x_1x_2$-plane, then $\alpha \cdot \beta = |\alpha| \, |\beta| \cos\theta$ where $\theta$ is the smaller positive angle between $\alpha$ and $\beta$.

EXERCISES:

1. Represent ( 3, -2 ) on a graph.



2. Represent ( 2,4,3 ) on a graph.




3. Find the magnitudes of the following vectors.

$\alpha = (2, 3, 4)$ $\quad \beta = (-1, 2)$ $\quad \gamma = (4, 0, -1)$

$$|\alpha| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{4+9+16} = \sqrt{29}$$

$$|\beta| = \sqrt{(-1)^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$$

$$|\gamma| = \sqrt{4^2 + 0^2 + (-1)^2} = \sqrt{16+1} = \sqrt{17}$$

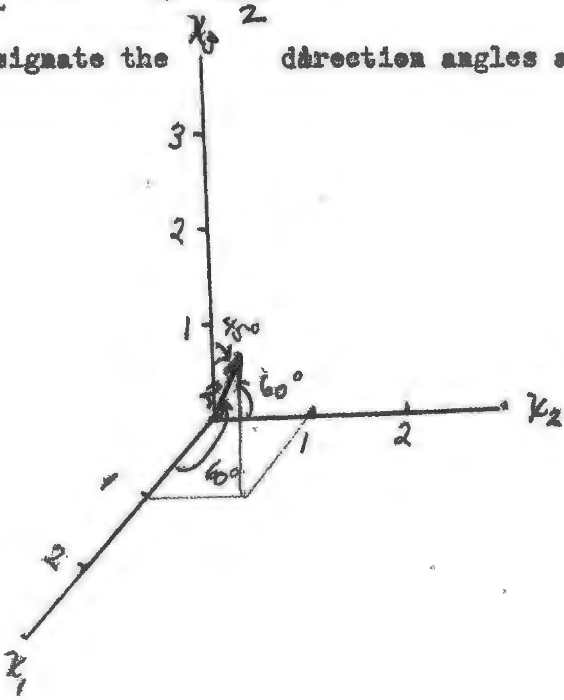
$\alpha = (1, 1, \sqrt{2})$

4. Find $|\alpha| = \sqrt{1+1+2} = \sqrt{4} = 2$

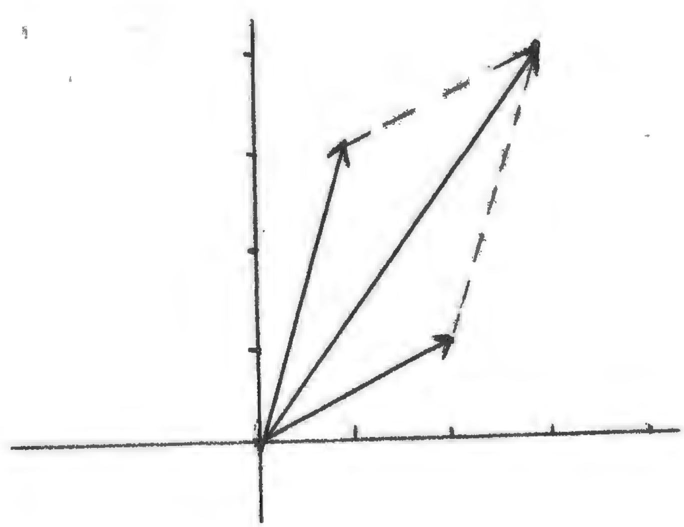
Find the direction angles of the vector.

$\cos\theta_1 = \frac{1}{2} \quad \cos\theta_2 = \frac{1}{2} \quad \cos\theta_3 = \frac{\sqrt{2}}{2}$

Graph the vector and designate the direction angles and the magnitude.




5. Add ( 2,1 ) and ( 1,3 ) graphically.



Evaluate each of the following products if a product exists.

1\. (a) $(2,4,0,7)\cdot(0,-1,6,2) = 0-4+0+14 = 10$

(b) $(2,6,3,0)\cdot(2,1,2) =$ the product does not exist because the vectors do not have the same dimensions.

2\. Solve for x:

$(x,1,2,0)\cdot(3,2,0,1) = 4$

$3x+2+0+0 = 4$

$3x = 2$

$x = 2/3$

3\. Find the cosine of the angle between these two vectors.

$\alpha = -2i+2j \qquad \beta = 4i+3j$

$\alpha\cdot\beta = -8+6 = -2 \qquad |\alpha| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2} \qquad |\beta| = \sqrt{16+9} = 5$

$\alpha\cdot\beta = |\alpha|\,|\beta|\cos\theta \qquad \cos\theta = \dfrac{-2}{2\sqrt{2}\cdot 5} = \dfrac{-2}{10\sqrt{2}} = \dfrac{-1}{5\sqrt{2}}$

Its being negative means it is an obtuse angle.

4\. Determine x so that the two vectors are perpendicular.

$\alpha = 2i+j \qquad \beta = xi+2j \qquad \cos 90^\circ = 0$

$\alpha\cdot\beta = 2x+2 = 0, \quad 2x = -2, \quad x = -1$

## CHAPTER THREE: MATHEMATICAL SYSTEMS.

A Mathematical system consists of a set of elements, at least one equivalence relation among these elements, at least one operation over these elements, and postulates concerning the elements, operations, and relations. A relation between two entities is a binary relation and designated by R where a R b means "a in the relation R to b." A relation R over a set A is an equivalence relation over set A if and only if the following properties are valid for all elements a, b, c of A:

(1) aRa ( reflexive property )

(2) if aRb, the bRa ( symmetric property )

(3) if aRb and bRc, then aRc ( transitive property )

The small letter "o" will be used to designate an operation used to combine two elements of a specified set. A binary operation "o" over a set S is a rule or procedure by which any two elements of S are combined to produce a unique third element which may or may not belong to S. If the third element always belongs to S, then we say S is closed under the operation "o". Four laws of operations are defined in this chapter. They are the commutative law, the associative law, the distributive law ( which includes two operations, a right and left distributive law ), and the cancellation law. If A,B, and C are matrices that are conformable for addition, the commutative and associative laws for matrix addition are valid. The associative law for matrix multiplication is valid and the left distributive law for matrix multiplication with respect to addition is valid. Multiplication of a matrix and a scalar is commutative. In general matrix operation laws for addition and multiplication differ from scalar. AB does not imply BA, AB=AC does not imply B=C, AB = 0 does not imply A=0 or B=0.

For n-dimensional vectors, the commutative and associative laws are valid for addition. The scalar product of two n-dimensional vectors is commutative, and the distributive laws for the scalar product with respect to addition are valid.

An element e in a set A, such that $a \circ e = a$ for every a in A, is called an identity element for the operation "o". $I_n$ is the identity matrix of order n defined formally as a square matrix of order n in which every entry on the main diagonal is 1 and all other entries are 0. Let e be an identity element for the operation o over the set A. If there exists an element q such that $a \circ q = q \circ a = e$ where e, a, and q belong to A, then q is called an inverse of a with respect to the operation o. The following table summarizes the laws that are valid for addition and multiplication of matrices and vectors.

MATRICES

| | addition | multiplication |
|---|---|---|
| COMMUTATIVE LAW | $A+B=B+A$ | no |
| ASSOCIATIVE LAW | $(A+B)*C = A+(B+C)$ | $(AB)C = A(BC)$ |
| DISTRIBUTIVE LAW | $A(B+C) = AB+AC$<br>$(A+B)C = AC+BC$ | |
| CANCELLATION LAW | $A+B = A+C$ implies $B=C$ | no |

VECTORS

| | addition | multiplication |
|---|---|---|
| COMMUTATIVE LAW | $\alpha+\beta=\beta+\alpha$ | $\alpha\cdot\beta=\beta\alpha$ |
| ASSOCIATIVE LAW | $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$ | Not Defined |
| DISTRIBUTIVE LAW | $\alpha\cdot(\beta+\gamma)=\alpha\cdot\beta+\alpha\cdot\gamma$<br>$(\alpha+\beta)\cdot\gamma=\alpha\cdot\gamma+\beta\gamma$ | |
| CANCELLATION LAW | $\alpha+\beta=\alpha+\gamma$ implies $\beta=\gamma$ | no |

A system of scalars ( or field ) is defined as a set S with at least two elements, a binary relation of equality over S, two binary operations ⊕ and ⊙ closed under S, and nine postulates. The postulates include the commutative law and associative law for both operations, an identity element e for both operations, an inverse for each element in S with respect to both operations except e with respect to ⊙ , and the distributive law with respect to ⊕ .

A ring is a system consisting of all the requirements for a field except for the commutative postulate for ⊙ , the identity element for ⊙ , and an inverse element for ⊙ . If you omit the inverse element for ⊙ and add the cancellation law for ⊙ you have a system called an integral domain.

A group is a system consisting of a set of elements S, a binary relation of equality over S, the binary operation o under S which is closed, and three postulates. The postulates include the associative law, an identity element for o, and an inverse in S for each element with respect to o. A group that also postulates o is commutative is called a commutative group or Abelian group.

EXERCISES:

1. Given the set of elements 1 and 0, the equivalence relation of equality, and the operations of ⊕ and • with the following postulates

(1) $0 \cdot 0 = 0$   (2) $1 \oplus 1 = 1$   (3) $1 \cdot 1 = 1$   (4) $0 \oplus 0 = 0$

(5) $1 \cdot 0 = 0 \cdot 1 = 0$   (6) $0 \oplus 1 = 1 \oplus 0 = 1$

prove the following theorems of binary Boolean algebra.

$x \oplus (x \cdot y) = x$

| x | y | ( x.y ) | x ⊕ ( x.y ) |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 |

Since the first and last column are equal the statement is true.

$x \oplus y = y \oplus x$

| x | y | x ⊕ y | y ⊕ x |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 |

Since the last two columns are equal the statement is true.

2. Let S be the set of all people in the world.

(a) Is "younger than" an equivalence relation among the elements of S?
No because a is not younger than a.

(b) Is "same age as" an equivalence relation among the elements of S?
Yes, because a is "same age as a, if a is same age as b then b is the same age as a, if a is same age as b, b is the same age as c, then a is the same age as c.

3. Suppose we are given the set of all positive integers and an operation

( * ) defined in the following way: a * b = a + 2b

Determine which of the laws mentioned in this section hold for this

operation: 4*3=4+6=10, 3*4=3+8=11; 4*(3*2)=4*7=4+14=18,
(4*3)*2=10*2=10+4=14; the cancellation law holds.

4. Suppose that $A=\begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$ $B=\begin{bmatrix} 4 & 7 \\ 3 & -5 \end{bmatrix}$ and $C=\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}$

(a) Verify the associative law for addition. $A+(B+C)=(A+B)+C$

$$\begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 15 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 3 & -4 \end{bmatrix} + \begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 12 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 2 & -5 \end{bmatrix}$$

(b) Verify the associative law for multiplication. $A(BC)=(AB)C$

$$\begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 25 \\ 11 & 29 \end{bmatrix} = \begin{bmatrix} -1 & 29 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -31 & -37 \\ 11 & 29 \end{bmatrix} = \begin{bmatrix} -31 & -37 \\ 11 & 29 \end{bmatrix}$$

(c) Verify the left distributive law for matrix multiplication with respect to addition. $A(B+C) = AB + AC$

$$\begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 6 & 15 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} -1 & 29 \\ 3 & -5 \end{bmatrix} + \begin{bmatrix} 7 & 19 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 48 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 6 & 48 \\ 2 & -6 \end{bmatrix}$$

(d) Verify the law which says c(AB) = (cA)B using c=3.

$$3\begin{bmatrix} -1 & 29 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 4 & 7 \\ 3 & -5 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 87 \\ 9 & -15 \end{bmatrix} = \begin{bmatrix} -3 & 87 \\ 9 & -15 \end{bmatrix}$$

5. Given the operations of + and (.) over the set of all even integers

(a) What is the identity for +, if any? I=0
(b) What is the identity for (.), if any? I=1
(c) What if the inverse of an even integer a for +, if any? -a
(d) What is the inverse of an even integer a for (.), if any? none.

6. Let an operation"∘" over the set of integers be defined as follows:

a ∘ b = a + b - 2.

(a) Which integer is the identity element for "∘"?
a ∘ x = a, a + x - 2 = a, x = 2.

(b) Which integer is the inverse of 3 with respect to "∘"?
3 ∘ x = 2, 3 + x - 2 = 2, x = 1.

7. Determine which of the following are examples of fields. If they are not fields tell which postulates do not hold. Let the operations be + and (.).

(a) the set of all integers; not a field because each element does not have an inverse.

(b) the set of all rational numbers; yes it is a field.

(c) the set of all pure imaginary numbers; no because it is not closed under (.), the is no identity element for either operation, and no inverse for either operation.

CHAPTER FOUR: SPECIAL MATRICES.

The transpose of a matrix A is a matrix which is formed by interchanging the rows and columns of A. The ith row of A becomes the ith column of the transpose of A, denoted by $A^t$. Some of the rules involving the transpose of a matrix are: $(A^t)^t = A$; $(A+B)^t = A^t+B^t$; $(AB)^t = B^tA^t$; where c is a scalar, $(cA)^t = cA^t$.

A matrix A is said to be symmetric when $A = A^t$. The product of a matrix and its transpose is symmetric. A matrix A is said to be skew-symmetric if $A = -A^t$. A must be square and each entry of the main diagonal must be zero. Any square matrix A is the sum of a skew-symmetric matrix and a symmetric matrix.

If the entries of a matrix A are complex numbers, the conjugate of A is the matrix $\overline{A}$ whose entries are the conjugates of the corresponding entries of A. $(\overline{A})^t$ is called the transposed conjugate or tranjugate matrix of A denoted by $A^*$. If $A = A^*$ it is said to be Hermitian. The matrix must be square and the main diagonal must be real numbers. A matrix is skew-Hermitian if $A = -A^*$. The rules for Hermitian matrices are: $(A^*)^* = A$; $(A + B)^* = A^* + B^*$; $(AB)^* = B^*A^*$; $(cA)^* = \overline{c}A^*$; $AA^*$ and $A^*A$ are Hermitian.

An echelon matrix is an m by n matrix constructed in the following manner: (a) Each of the first K rows has some nonzero entries. The entries are all zeros in the remaining m-k rows ( $1 \leq k \leq m$ ). (b) The first nonzero entry in each of the first k rows is 1. (c) In any one of the first k rows, the number of zeros preceding the first nonzero entry is smaller than it is in the next following row.

A submatrix of a matrix A is the rectangular array that remains if certain rows or columns ( or both ) of A are deleted. For a system of linear equations AX=B, A is called the coefficient matrix, and $[A \mid B]$ is the augmented matrix.

EXERCISES:

1. Find the transpose of each of the following matrices:

(a) $\begin{bmatrix} 2 & -3 \\ 5 & 6 \end{bmatrix}$ transpose $= \begin{bmatrix} 2 & 5 \\ -3 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 4 & -2 \end{bmatrix}^T = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$

2. Show that $( ABC )^t = C^tB^tA^t$.

$( ABC )^t = (BC)^tA^t = C^tB^tA^t$.

3. $(A^t + 2B^t + C)^t$ Simplify: $(A^t + 2B^t + C)^t = (A^t)^t + (2B^t)^t + C^t = A + 2(B^t)^t + C^t = A + 2B + C^t$

4. Find the conjugate matrix and the tranjugate matrix of the following:

$A = \begin{bmatrix} i & 2-i & 2 \\ 0 & 3i & 4 \end{bmatrix}$ $\bar{A} = \begin{bmatrix} -i & 2+i & 2 \\ 0 & -3i & 4 \end{bmatrix}$ $A^* = \begin{bmatrix} -i & 0 \\ 2+i & -3i \\ 2 & 4 \end{bmatrix}$

5. Find AB using the indicated partitioning.

$A = \left[\begin{array}{cc} 0 & 2 \\ \hline 4 & 3 \\ 2 & 1 \end{array}\right]$ $B = \left[\begin{array}{c|c} 1 & 2 \\ 0 & 4 \end{array}\right]$ $AB = \left[\begin{array}{c|c} [0\ 2]\begin{bmatrix}1\\0\end{bmatrix} & [0\ 2]\begin{bmatrix}2\\4\end{bmatrix} \\ \hline \begin{bmatrix}4 & 3\\2 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} & \begin{bmatrix}4 & 3\\2 & 1\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix} \end{array}\right] = \left[\begin{array}{c|c} 0 & 8 \\ \hline 4 & 20 \\ 2 & 8 \end{array}\right]$

6. Write the augmented and coefficient matrix for:

$$\begin{cases} X_1 - X_2 + X_3 = 4 \\ 2X_1 \qquad + X_3 = 2 \\ \qquad X_2 + 2X_3 = 1 \end{cases} \qquad \underset{\text{Coefficient}}{\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}}, \qquad \underset{\text{Augmented}}{\begin{bmatrix} 1 & -1 & 1 & 4 \\ 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}}$$

CHAPTER FIVE: DETERMINANTS.

The determinant of $A = [a_{ij}]_n$ is the sum of all terms of the form $(-1)^t a_{1j1}\, a_{2j2} \ldots a_{njn}$. where the second subscripts assume all possible arrangements in which each column is represented exactly once in each term of the sum, and the exponet, t, is the number of interchanges necessary to bring the second subscripts into natural order ( that is 1, 2, 3, . . .n ). The minor of an entry $a_{ij}$ of a square matrix A is the determinant of the submatrix of A obtained by deleting the ith row and jth column. The cofactor of an entry $a_{ij}$ of a square matrix A is the product of the minor and $(-1)^{i+j}$. This cofactor is denoted by $A_{ij}$. The determinants of a matrix and its transpose are equal. The determinant of a matrix with two identical parallel lines is zero. The determinant of the product of two square matrices of the same order is equal to the product of the determinants of the two matrices.

According to Cramer's Rule if the det $A \neq 0$, then the system AX=B has exactly one solution; This solution is $x_j = \dfrac{\det({}^jA)}{\det A}$, $j = 1,2, \ldots n$.

The rank of a matrix is the order of the largest square submatrix whose determinant is not zero. When all of the entries are zero the rank is zero.

EXERCISES:

1. Evaluate by definition:

(a) $\begin{vmatrix} 2 & 3 \\ -6 & 1 \end{vmatrix} = (-1)^0 2\cdot 1 + (-1)^1(3)\cdot(-6) = 2+18 = 20$

(b) $\begin{vmatrix} 0 & -2 \\ -1 & 4 \end{vmatrix} = (-1)^0 0\cdot 4 + (-1)^1(-2)(-1) = 0-2 = -2$

2. Given $A = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 2 & 2 \\ -1 & 4 & 0 \end{bmatrix}$

(a) Expand det A about the first column.

$$|A| = 2(-1)^2\begin{vmatrix}2&2\\4&0\end{vmatrix} + 3(-1)^3\begin{vmatrix}3&0\\4&0\end{vmatrix} + (-1)(-1)^4\begin{vmatrix}3&0\\2&2\end{vmatrix} = 2(-8)-3(0)-6 = -16-6 = -22$$

(b) Expand det A about the third row.

$$|A| = -1(-1)^4\begin{vmatrix}3&0\\2&2\end{vmatrix} + 4(-1)^5\begin{vmatrix}2&0\\3&2\end{vmatrix} + 0 = -6-4(4) = -6-16 = -22$$

(c) Expand det A about third column.

$$|A| = 0 + 2(-1)^5\begin{vmatrix}2&3\\-1&4\end{vmatrix} = -2(8+3) = -2(11) = -22$$

(d) What is the cofactor of the entry in the third row and second column? $(-1)^5\begin{vmatrix}2&0\\3&2\end{vmatrix} = -(4) = -4$

(e) What is the minor of the entry in the first row and second column?

$$\begin{vmatrix}3&2\\-1&0\end{vmatrix} = (0+2) = 2$$

3. Evaluate det A by row or column expansion.

$$A = \begin{bmatrix}3&0&2&0\\3&0&0&2\\0&2&3&0\\2&0&1&3\end{bmatrix} \quad \det A = 0+0+2(-1)^5\begin{vmatrix}3&2&0\\3&0&2\\2&1&3\end{vmatrix} + 0 = -2\begin{vmatrix}3&2&0\\3&0&2\\2&1&3\end{vmatrix} =$$

$$-2\left\{2(-1)^2\begin{vmatrix}3&0\\2&3\end{vmatrix} + 0 + (-1)^5\begin{vmatrix}3&0\\3&2\end{vmatrix}\right\} = -2\{2(9)-5(6)\} = -2(18-30) = 24$$

4. Verify:

$$A = \begin{vmatrix}0&-1&1&1\\0&2&-1&-1\\2&-3&2&4\\2&-2&1&3\end{vmatrix} = 0$$

$$\det A = 0+0+2(-1)^4\begin{vmatrix}-1&1&1\\2&-1&-1\\-2&1&3\end{vmatrix} + 2(-1)^5\begin{vmatrix}-1&1&1\\2&-1&-1\\-3&2&4\end{vmatrix} = 2\left\{-1(-1)^2\begin{vmatrix}-1&-1\\1&3\end{vmatrix} + (-1)^3\begin{vmatrix}2&-1\\-2&3\end{vmatrix} +\right.$$

$$\left.(-1)^4\begin{vmatrix}2&-1\\-2&1\end{vmatrix}\right\} - 2\left\{-1(-1)^2\begin{vmatrix}-1&-1\\2&4\end{vmatrix} + (-1)^3\begin{vmatrix}2&-1\\-3&4\end{vmatrix} + (-1)^4\begin{vmatrix}2&-1\\-3&2\end{vmatrix}\right\} =$$

$$2\{-(-3+1)-(6-2)+(2-2)\} - 2\{-(-4+2)-(8-3)+(4-3)\} =$$

$$2(2-4) - 2(2-5+1) = 2(-2) - 2(-2) = 0$$

5. In each of the following systems find y by Cramer's rule and then find the other unknowns by substitution.

(a) $\begin{cases} x+y=5 \\ 2x-y=7 \end{cases}$ $\quad y = \dfrac{\begin{vmatrix} 1 & 5 \\ 2 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}} = \dfrac{(7-10)}{(-1-2)} = \dfrac{-3}{-3} = 1 \qquad x = 5-1 = 4,\ y = 1$

(b) $\begin{cases} x+y+z=0 \\ -y+2z=4 \\ 2x+2y \phantom{+2z} =1 \end{cases}$ $\quad y = \dfrac{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 4 & 2 \\ 2 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 2 & 2 & 0 \end{vmatrix}} = \dfrac{(-2)+(-8)}{(-4)+2(2+1)} = \dfrac{-10}{2} = -5$

6. Find the ranks of :

(a) $A = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 5 \\ 3 & -1 & 6 \end{bmatrix}$ $\quad \det \text{ of } A = (6+5)-2(24-15)$ [redacted] $11-18+7=0$

$\begin{vmatrix} 4 & 1 \\ 3 & -1 \end{vmatrix} = 4(-1)-3 = -7 \qquad \text{Rank} = 2$

(b) $A = \begin{bmatrix} 2 & 4 \\ -4 & -8 \\ 1 & 2 \end{bmatrix}$ [redacted] $= 2(-8)-4(-4) = -16+16 = 0$

$\begin{vmatrix} -4 & -8 \\ 1 & 2 \end{vmatrix} = -4(2)+8 = 0$

$\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 2(2)-4 = 0 \qquad \text{RANK} = 1$

8. Find the rank of the augmented matrix and the coefficient matrix of the system:

$\begin{cases} x+y=6 \\ x+3y=4 \\ 2x-y=2 \end{cases}$ $\quad A = \begin{bmatrix} 1 & 1 & 6 \\ 1 & 3 & 4 \\ 2 & -1 & 2 \end{bmatrix}$ $\quad \det A = (3-2)-(2+6)+2(4-18) = 1-8+28 = 21$

$\text{RANK} = 3$

$C = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 2 & -1 \end{bmatrix} \quad \det \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 3-1 = 2$

$\text{Rank} = 2$

## CHAPTER SIX: THE INVERSE MATRIX.

An inverse, $A^{-1}$, of a given square matrix A, if it exists, is a square matrix such that $AA^{-1}=A^{-1}A=I$ where I is the identity matrix whose order is the same as that of A. A square matrix A is said to be singular if det $A=0$ and nonsingular if det $A\neq 0$. If A is a square matrix whose order is two or greater the cofactor matrix of A, designated by cofA, is the matrix of order n whose entry in row i and column j is the cofactor of the corresponding element in A. The adjoint matrix, designated adjA, of a square matrix A is the transpose of cofA. For a square matrix A, $A^{-1}$ exists if and only if A is nonsingular. More over if $A^{-1}$ exists then $A^{-1}=\frac{1}{\det A}\,\mathrm{adj}A$ .

Several theorems involving these properties were listed. If A is nonsingular, $(A^t)^{-1}=(A^{-1})^t$, and $(A^{-1})^{-1}=A$. If $AB=0$ and A, B are square of order n then $A=0$, or $B=0$ or both A and B are singular. If A, B, and C are square of order n and A is nonsingular, then $AB=AC$ implies $B=C$. If the system $AX=B$, where A is nonsingular, has a unique solution, the solution is $X=A^{-1}B$. This is used only when the number of equations equals the number of unknowns and A is nonsingular.

EXERCISES:

1. Calculate the adjoint matrices for the following matrices:

(a) $A=\begin{bmatrix}2 & 3\\ -4 & 1\end{bmatrix}$ $\mathrm{adj}A=(\mathrm{cof}\,A)^T=\begin{bmatrix}1 & 4\\ 3 & 2\end{bmatrix}^T=\begin{bmatrix}1 & 3\\ 4 & 2\end{bmatrix}$

(b) $B=\begin{bmatrix}0 & 1 & -1\\ 0 & -2 & 2\\ -1 & 1 & 0\end{bmatrix}$ $\mathrm{adj}B=(\mathrm{cof}\,B)^T=\begin{bmatrix}-4 & -2 & -2\\ -2 & -1 & -2\\ -2 & 0 & 0\end{bmatrix}^T=\begin{bmatrix}-4 & -2 & -2\\ -2 & -1 & 0\\ -2 & -2 & 0\end{bmatrix}$

2. Determine whether the matrices in the first problem are nonsingular or singular.

det $A = 2+4=6$ nonsingular

det $B = -(2-2)=0$ singular

3. Given the matrix A show that det(adjA)= $(\det A)^2$. $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 3 & 0 & 2 \end{bmatrix}$

Cof $A = \begin{bmatrix} 4 & 3 & -6 \\ -2 & 4 & 3 \\ 1 & -2 & 4 \end{bmatrix}$ adj $A = \begin{bmatrix} 4 & -2 & 1 \\ 3 & 4 & -2 \\ -6 & 3 & 4 \end{bmatrix}$

$\det A(\operatorname{adj} A) = 4(16+6) + 2(12-12) + (9+24) = 4(22)+33 = 88+33 = 121$

$(\det A)^2 = 121 \qquad 121 = 121$

$\therefore \det(\operatorname{adj} A) = (\det A)^2$

4. Calculate the inverse of the following matrix and check by $AA^{-1}=A^{-1}A=I$.

$A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ Cof $A = \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}$ adj $A = \begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix}$ det $A = 2(3)+4=10$

$A^{-1} = \begin{bmatrix} 3/10 & 1/10 \\ -4/10 & 2/10 \end{bmatrix}$ $AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^{-1}A = I$

5. Using the following two matrices illustrate $(AB)^{-1}=B^{-1}A^{-1}$

$A = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

Cof $A = \begin{bmatrix} 0 & -2 \\ 3 & 1 \end{bmatrix}$ adj $A \begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix}$ det $A = 6$ Cof $B = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$ adj $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$ det $B = 1$

$A^{-1} = \begin{bmatrix} 0 & 3/6 \\ -2/6 & 1/6 \end{bmatrix}$ $B^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & -1 \\ 6 & 4 \end{bmatrix}$ Cof$(AB) = \begin{bmatrix} 4 & -6 \\ 1 & 0 \end{bmatrix}$ adj$(AB) = \begin{bmatrix} 4 & 1 \\ -6 & 0 \end{bmatrix}$ det$(AB) = 6$

$(AB)^{-1} = \begin{bmatrix} 4/6 & 1/6 \\ -1 & 0 \end{bmatrix}$ $B^{-1}A^{-1} = \begin{bmatrix} 4/6 & 1/6 \\ -1 & 0 \end{bmatrix} = (AB)^{-1}$

6. Solve the following systems of equations.

$\begin{cases} X_1 + 3X_2 = 4 \\ 2X_1 - 2X_2 = 6 \end{cases}$ $A = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}$ $c = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

Cof $A = \begin{bmatrix} -2 & -2 \\ -3 & 1 \end{bmatrix}$ adj $A = \begin{bmatrix} -2 & -3 \\ -2 & 1 \end{bmatrix}$ det $A = -8$ $A^{-1} = \begin{bmatrix} 1/4 & 3/8 \\ 1/4 & -1/8 \end{bmatrix}$

$X = \begin{bmatrix} 1/4 & 3/8 \\ 1/4 & -1/8 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 3\,1/4 \\ 1/4 \end{bmatrix}$

$X_1 = 3\,1/4$

$X_2 = 1/4$

CHAPTER SEVEN: ELEMENTARY MATRIX TRANSFORMATIONS.

Three operations are defined for matrices. They are: the interchange of any two rows, to multiply any row by a nonzero scalar, to add to any row a scalar multiple of another row. These are called elementary tow operations. Elementary column operations are defined by replacing the word "row" by "column". When a matrix has been reduced by elementary operations to one of the forms: $\left[\begin{array}{c} I_r \\ \hline 0 \end{array}\right]$ $\left[\begin{array}{c|c} I_r & 0 \end{array}\right]$ $\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$ or $I_r$ ( r= rank)

we say that it has been reduced to normal form.

Matrices are equivalent to another when they have the same rank. A matrix is an elementary matrix when it can be obtained from the identity matrix I by an elementary operation. If two matrices A and B are equivalent, then there exist two nonsingular matrices P and Q, such that A=PNQ. If A is nonsingular and if the matrix $\left[\begin{array}{c|c} A & I \end{array}\right]$ is transformed to the equivalent matrix $\left[\begin{array}{c|c} I & P \end{array}\right]$ by elementary row operations, then P is the inverse of A.

EXERCISES:

1. Using elementary row operations, write an echelon matrix which is row equivalent to the given matrix.

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 1 & 8 & 0 & 2 \\ 2 & 0 & 4 & 2 \end{bmatrix} \underset{-R_1+R_2}{\sim} \begin{bmatrix} 1 & 4 & 3 & 2 \\ 0 & 4 & -3 & 0 \\ 2 & 0 & 4 & 2 \end{bmatrix} \underset{-2R_1+R_3}{\sim} \begin{bmatrix} 1 & 4 & 3 & 2 \\ 0 & 4 & -3 & 0 \\ 0 & -8 & -2 & 2 \end{bmatrix}$$

$$\underset{2R_2+R_3}{\sim} \begin{bmatrix} 1 & 4 & 3 & 2 \\ 0 & 4 & -3 & 0 \\ 0 & 0 & -8 & -2 \end{bmatrix} \underset{\frac{1}{4}R_2}{\sim} \begin{bmatrix} 1 & 4 & 3 & 2 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & -8 & -2 \end{bmatrix} \underset{-1/8\,R_3}{\sim} \begin{bmatrix} 1 & 4 & 3 & 2 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 1 & 1/4 \end{bmatrix}$$

2. Write the augmented matrix for the following system and transform it into an echelon matrix by row operations. Then write the system for which this echelon matrix is the augmented matrix and find the solution.

$$\begin{cases} 2x + y = 6 \\ 2x + 4y = 9 \end{cases} \qquad \begin{bmatrix} 2 & 1 & 6 \\ 2 & 4 & 9 \end{bmatrix} \underset{-R_1+R_2}{\sim} \begin{bmatrix} 2 & 1 & 6 \\ 0 & 3 & 3 \end{bmatrix} \underset{\frac{1}{2}R_1}{\sim} \begin{bmatrix} 1 & 1/2 & 3 \\ 0 & 3 & 3 \end{bmatrix} \underset{1/3\,R_2}{\sim} \begin{bmatrix} 1 & 1/2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{cases} x + \frac{1}{2}y = 3 \\ y = 1 \end{cases} \qquad y = 1 \quad x = 5/2$$

3. Without interchanging rows, change $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ to $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ in four steps using elementary row operations.

$$\underset{-R_1+R_2}{\sim} \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \underset{R_2+R_1}{\sim} \begin{bmatrix} 3 & 4 \\ 2 & 2 \end{bmatrix} \underset{-R_1+R_2}{\sim} \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix} \underset{-R_2}{\sim} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

4. Find the rank of the given matrix by reducing it to normal form:

$$\begin{bmatrix} 3 & 2 & -1 \\ 7 & 8 & 0 \\ 4 & 6 & 1 \end{bmatrix} \underset{C_1 \leftrightarrow C_3}{\sim} \begin{bmatrix} -1 & 2 & 3 \\ 0 & 8 & 7 \\ 1 & 6 & 4 \end{bmatrix} \underset{R_1+R_3}{\sim} \begin{bmatrix} -1 & 2 & 3 \\ 0 & 8 & 7 \\ 0 & 8 & 7 \end{bmatrix} \underset{-R_1+R_3}{\sim} \begin{bmatrix} -1 & 2 & 3 \\ 0 & 8 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underset{-4R_1+R_2}{\sim} \begin{bmatrix} -1 & 2 & 3 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix} \underset{\substack{2R_1+R_2 \\ 2C_1+C_2 \\ 3C_1+C_3}}{\sim} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \underset{-C_1}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \underset{C_2 \leftrightarrow C_3}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

RANK = 2

5. Write the elementary matrix E which performs the indicated elementary transformation on A= $\begin{bmatrix} -3 & 2 & -1 \\ 4 & 0 & 1 \end{bmatrix}$ Then multiply to obtain the desired transformation.

(a) Interchange the first and second rows.

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad EA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 & -1 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 1 \\ -3 & 2 & -1 \end{bmatrix}$$

(b) Interchange the first and third columns.

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad AE = \begin{bmatrix} -3 & 2 & -1 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -3 \\ 1 & 0 & 4 \end{bmatrix}$$

6. Use the following matrices and find matrices P and Q such that A=PBQ. $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$B \underset{-3R_1+R_2}{\sim} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \underset{-2C_1+C_2}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \underset{-\frac{1}{2}R_2}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A \underset{\substack{-3R_1+R_2 \\ -\frac{1}{2}R_2}}{\sim} \begin{bmatrix} 1 & 0 \\ 3/2 & -1/2 \end{bmatrix} = P \qquad A = PBQ = \begin{bmatrix} 1 & 0 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$A \underset{-2C_1+C_2}{\sim} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = Q \qquad = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

7. Find the inverse matrix by elementary operations.

(a) $A = \begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix}$ $\quad [A \mid I] = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{array}\right] \underset{-4R_1+R_2}{\sim} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -12 & -4 & 1 \end{array}\right]$

$\underset{-\frac{1}{12}R_2}{\sim} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 1/3 & -1/12 \end{array}\right] \underset{-3R_2+R_1}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 1/3 & -1/12 \end{array}\right]$

$A^{-1} = \begin{bmatrix} 0 & 1/4 \\ 1/3 & -1/12 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $\quad [A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \underset{-3R_3+R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

$\underset{\substack{R_1 \leftrightarrow R_3 \\ -R_1 + R_3}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -1 \end{bmatrix}$

CHAPTER EIGHT: SYSTEMS OF LINEAR EQUATIONS.

If AX=B is a matrix equation representing a system of equations and B=0, the system is called homogeneous. If B≠0, the system is heterogeneous. The system of m linear equations in n unknowns may: (1) have no solution and therefore be inconsistent, (2) have exactly one ( unique ) solution, or (3) an infinite number of solutions and therefore be consistent. A system of linear equations AX=B is consistent if and only if the rank of the augmented matrix is equal to the rank of the coefficient matrix. A consistent system of linear equations AX=B has a unique solution if and only if r=n,( where r is the common value called the rank of the system). Homogeneous equations are always consistent because the ranks of the coefficient and augmented matrices are equal.

There is a means of determing whether r exists and if so whether r=n while at the same time working toward a solution. This is called the echelon method. The first step is to transform the augmented matrix to an echelon matrix by elementary operations. The echelon matrix may be used to determine r and transforming it back to a system of equations will make substitution to find the solutions easy. If the echelon matrix reveals that there are qeors everywhere in the last row except for the last column, the system will be inconsistent. With systems that are consistent but have more than one solution one can find the complete solution by expressing each unknown in terms of one of the others. A particular solution may be derived from the complete solution by substituting values for the parameter. The Gauss-Jordon Elimination Method is sometimes used where the echelon matrix is further transformed to create a submatrix $I_r$. This is augmented and then substitution is used.

A consistent system of linear equations of rank r can be solved for r unknowns say $x_{i1}, x_{i2}, \ldots x_{ir}$ in terms of the remaining m-r unknowns, if and only if the submatrix of coefficients of $x_{i1}, x_{i2}, \ldots x_{ir}$ has rank r. In a consistent system with more unknowns than equations and the complete solution is found a basic solution can be found by assigning the parameters the value zero. The unknowns or variables not serving as parameters in a basic solution are called basic variables.

EXERCISES:

1.) Determine whether the following systems are inconsistent, consistent with a unique solution, or consistent with an infinite number of solutions.

(a) $\begin{cases} x_1 + 3x_2 + x_3 = 4 \\ x_1 + x_2 - x_3 = 1 \\ 2x_1 + 4x_2 = 0 \end{cases}$ $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & -1 \\ 2 & 4 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 1 & 1 & -1 & 1 \\ 2 & 4 & 0 & 0 \end{bmatrix}$

$|A| = 1(4) - 1(-4) + 2(-3-1) = 0$

$\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = 1(1) - 1(3) = -2$ Rank $A = 2$

$\begin{vmatrix} 1 & 1 & 4 \\ 1 & -1 & 1 \\ 2 & 0 & 0 \end{vmatrix} = 1(0) - 1(0) + 2(1+4) = 10$ RANK $C = 10$ $2 \neq 10$

Therefore, the system is inconsistent.

(b) $\begin{cases} x_1 - x_2 + 6 = 0 \\ x_1 + 2x_2 - 5 = 0 \\ 3x_1 + 3x_2 - 4 = 0 \end{cases}$ $A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 3 & 3 \end{bmatrix}$ $B = \begin{bmatrix} -6 \\ 5 \\ 4 \end{bmatrix}$ $[A \mid B] = \begin{bmatrix} 1 & -1 & -6 \\ 1 & 2 & 5 \\ 3 & 3 & 4 \end{bmatrix}$

$\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 2+1 = 3$ RANK $A = 2$

$\det [A \mid B] = 1(8-15) - 1(-4+18) + 3(-5+12)$
$= -7 - 14 + 21 = 0$

RANK $[A \mid B] = 2$

Therefore, the system is consistent and has a unique solution, r=2.

2. Verify the following system is inconsistent and then sketch the graph of the equations of the system.

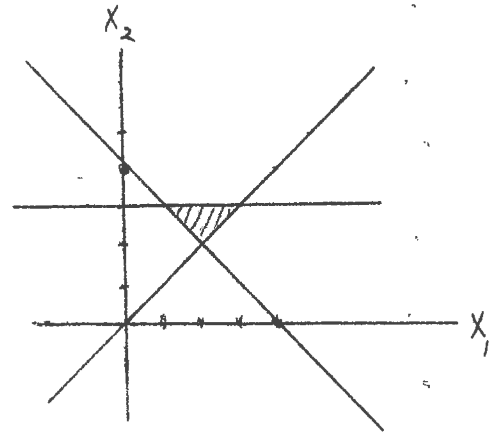
$$\begin{cases} X_1 + X_2 = 4 \\ X_1 - X_2 = 0 \\ \quad X_2 = 3 \end{cases} \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad [A \mid B] = \begin{bmatrix} 1 & 1 & 4 \\ 1 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1 = -2 \quad \text{Rank } A = -2$$

$$|A \mid B| = 1(-3) - 1(3-4) + 0 = -3+1 = 2$$

$$\text{RANK } [A \mid B] = 3, \quad 2 \neq 3$$

Therefore, the system is inconsistent.




3. A trucking company owns three types of trucks, numbered 1,2,and 3, which are equipped to haul three different types of machines per load according to the following chart.

| | No. 1 | no. 2 | no. 3 |
|---|---|---|---|
| Machines A | 1 | 1 | 1 |
| Machines B | 0 | 1 | 2 |
| Machines C | 2 | 1 | 1 |

How many trucks of each type should be sent to haul exactly 12 of the type A machines, 10 of the type B machines, and 16 of the type C machines?

$$\begin{cases} X_1 + X_2 + X_3 = 12 \\ \quad X_2 + 2X_3 = 10 \\ 2X_1 + X_2 + X_3 = 16 \end{cases} \qquad \begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & 2 & 10 \\ 2 & 1 & 1 & 16 \end{bmatrix} \underset{-2R_1+R_3}{\sim} \begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & 2 & 10 \\ 0 & -1 & -1 & -8 \end{bmatrix}$$

$$\underset{R_2+R_3}{\sim} \begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & 2 & 10 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad \begin{cases} X_1 + X_2 + X_3 = 12 \\ \quad X_2 + 2X_3 = 10 \\ \quad\quad X_3 = 2 \end{cases}$$

$X_3 = 2, \; X_2 = 6, \; X_1 = 4$

$X_1 = \#1 \quad X_2 = \#2 \quad X_3 = \#3$

4. By the Gauss-Jordan elimination method or the echelon method, determine whether the system is consistent and has more than one solution.

$$\begin{cases} x_1 + x_2 - x_3 = 4 \\ x_1 - x_2 + x_3 = 2 \end{cases} \qquad \begin{bmatrix} 1 & 1 & -1 & 4 \\ 1 & -1 & 1 & 2 \end{bmatrix} \underset{-R_1+R_2}{\sim} \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & -2 & 2 & -2 \end{bmatrix}$$

$$\underset{-\frac{1}{2}R_2}{\sim} \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & -1 & 1 \end{bmatrix} \underset{-R_2+R_1}{\sim} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

$$\begin{cases} x_1 = 3 \\ x_2 - x_3 = 1 \end{cases} \qquad \begin{cases} x_1 = 3 + 0x_3 \\ x_2 = x_3 + 1 \end{cases}$$

5. In the following consistent system, can we solve for $x_1$ and $x_2$ in terms of $x_3$?

$$\begin{cases} x_1 + x_2 + x_3 = 4 \\ -x_1 - x_2 = 2 \\ 2x_1 + 2x_2 + 2x_3 = 8 \end{cases} \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 2 & 2 & 2 \end{bmatrix} \qquad |A| = 1(-2) + 0 + 2(1) = 0$$

$$\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = -1 + 1 = 0$$

$$\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = 0 + 1 = 1$$

$$r = 2 \qquad C = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 2 & 2 \end{bmatrix} = \text{coefficient matrix for } x_1, x_2.$$

$$|C| = (-2) + 2 = 0$$

Since the rank of the coefficient matrix does not equal r we can not solve the equations in terms of $x_3$.

CHAPTER NINE: VECTOR SPACES.

A set of elements forms a vector space if: any two of the elements can be added in a unique way and their sum is also an element of the set, any element b of the set can be multiplied by an arbitrary scalar s to produce a unique element sb of the set, the laws listed below are satisfied. If a,b,c represent arbitrary elements of the set and r,s represent arbitrary scalars, a+b=b+a, a+(b+c)=(a+b)+c, and there exists an element 0 in the set such that a+0=a, there exists an element -a in the set such that a+(-a)=0, r(a+b)=ra+rb, (r+s)a=ra+sa, r(sa=(rs)a, there exists a scalar 1, known as the identity element for multiplication such that 1.a=a.

If $\alpha_1, \alpha_2, \cdots \alpha_n$ are vectors and $k_1, k_2 \ldots k_n$ are scalars, the vector $\gamma = k_1\alpha_1, k_2\alpha_2, \cdots k_n\alpha_n$ is said to be a linear combination of the vectors $\alpha_1, \alpha_2 \cdots\cdots \alpha_n$. A set of vectors $\alpha_1, \alpha_2 \cdots \alpha_n$ is said to span a vector space provided that they belong to the vector space and that every vector in the space can be expressed as a linear combination of them. The n vectors $\alpha_1, \alpha_2, \cdots \alpha_n$ are said to be linearly dependent if there exists a set of scalars $k_1$, $k_2 \ldots k_n$ not all zero, such that $k_1\alpha_1 + k_2\alpha_2 + \cdots k_n\alpha_n = 0$. A set of vectors $\alpha_1, \alpha_2 \cdots \alpha_n$ which is not linearly dependent is said to be linearly independent. That is if $\alpha_1, \alpha_2 \cdots \alpha_n$ are linearly independent and if $k_1\alpha_1 + k_2\alpha_2 \cdots k_n\alpha_n = 0$ then $k_1 = k_2 = \ldots k_n = 0$. If the number of vectors exceeds the number of components of the vectors involved, then the set of vectors must be linearly dependent. In order for a set of vectors to span n-dimensional space the set must contain at least n-vectors. A vector space is said to be generated by a certain set of vectors if the vector space consists of all of the linear combinations of vectors of that set.

EXERCISES.

1. Determine whether the following sets of vectors are linearly dependent or independent.

(a) $(2,-1), (-4,2)$ $\quad K_1(2,-1)+K_2(-4,2)=0$ $\quad \begin{cases} 2K_1-4K_2=0 \\ -K_1+2K_2=0 \end{cases}$

$$\begin{bmatrix} 2 & -4 & 0 \\ -1 & 2 & 0 \end{bmatrix} \underset{R_2+R_1}{\sim} \begin{bmatrix} 1 & -2 & 0 \\ -1 & 2 & 0 \end{bmatrix} \underset{R_1+R_2}{\sim} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \{K_1-2K_2=0$$

They are linearly dependent.

(b) $(2,1,0), (1,3,2), (0,9,1)$ $\quad K_1(2,1,0)+K_2(1,3,2)+K_3(0,9,1)=0$

$$\begin{cases} 2K_1+K_2 = 0 \\ K_1+3K_2+9K_3=0 \\ 2K_2+K_3=0 \end{cases} \quad \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 3 & 9 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \underset{-R_2+R_1}{\sim} \begin{bmatrix} 1 & -2 & -9 & 0 \\ 1 & 3 & 9 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

$$\underset{-R_1+R_2}{\sim} \begin{bmatrix} 1 & -2 & -9 & 0 \\ 0 & 5 & 18 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \underset{\frac{1}{5}R_2}{\sim} \begin{bmatrix} 1 & -2 & -9 & 0 \\ 0 & 1 & 18/5 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \underset{-2R_2+R_3}{\sim}$$

$$\begin{bmatrix} 1 & -2 & -9 & 0 \\ 0 & 1 & 18/5 & 0 \\ 0 & 0 & -31/5 & 0 \end{bmatrix} \underset{-5/31\, R_3}{\sim} \begin{bmatrix} 1 & -2 & -9 & 0 \\ 0 & 1 & 18/5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} K_1-2K_2-9K_3=0 \\ K_2+18/5K_3=0 \\ K_3=0 \end{cases} \qquad \begin{cases} K_1=0 \\ K_2=0 \\ K_3=0 \end{cases}$$

They are linearly independent.

2. Which of the following sets of vectors are not a basis of the vector space consisting of all three-dimensional vectors?

(a) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad \begin{matrix} X_1+2X_2=0 \\ 0X_1+0X_2=0 \\ X_1+X_2+X_3=0 \end{matrix}$ $\quad \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

$$\underset{R_2 \leftrightarrow R_3}{\sim} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{-R_1+R_2}{\sim} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{-R_2}{\sim} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 + 2x_2 = 0 \\ \qquad x_2 - x_3 = 0 \end{cases} \qquad \begin{cases} x_1 = -2x_3 \\ x_2 = x_3 \end{cases}$$

This system is not a basis because it is not linearly independent nor span the vector space.

(b) $\begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \begin{matrix} 2x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \\ 4x_1 + x_2 = 0 \end{matrix}$ $\quad \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 4 & 1 & 0 \end{bmatrix} \underset{\frac{1}{2}R_1}{\sim} \begin{bmatrix} 1 & 1/2 & 0 \\ 1 & 1 & 0 \\ 4 & 1 & 0 \end{bmatrix} \underset{-R_1+R_2}{\sim}$

$$\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 0 \\ 4 & 1 & 0 \end{bmatrix} \underset{2R_2}{\sim} \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 0 \\ 4 & 1 & 0 \end{bmatrix} \underset{-4R_1+R_3}{\sim} \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \underset{R_2+R_3}{\sim}$$

$$\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{cases} x_1 + \frac{1}{2}x_2 = 0 \\ \qquad x_2 = 0 \end{cases} \qquad x_1 = x_2 = 0$$

This system is not a basis because it does not span the vector space.

(c) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\begin{cases} x_1 + 3x_2 + x_3 = 0 \\ 2x_1 + x_2 \qquad = 0 \\ x_1 \qquad\qquad = 0 \end{cases}$ $\quad \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$$\underset{-2R_3+R_2}{\sim} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \underset{-R_1+R_3}{\sim} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & -1 & 0 \end{bmatrix} \underset{-\frac{1}{3}R_3}{\sim} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1/3 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 + 3x_2 + x_3 = 0 \\ \qquad x_2 \qquad = 0 \\ \qquad x_2 + \frac{1}{3}x_3 = 0 \end{cases} \qquad \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$

This system is a basis because it is linearly independent and spans the vector space.

## CHAPTER TEN: LINEAR TRANSFORMATIONS.

The transformation of the elements of one set into the elements of another set is sometimes called a mapping. In mapping the elements of set U into the elements of a set V, the set U is called the domain of the mapping. The set V is called the range of the mapping. A transformation of vectors is called a matrix transformation if it can be expressed by multiplying the vector by a matrix. A transformation T of a vector space V into V is said to be a linear transformation if $T(\alpha+\beta)=T(\alpha)+T(\beta)$, and $T(K\alpha)=KT(\alpha)$. Every matrix transformation is a linear transformation.

EXERCISES:

1. Determine the matrix A such that $\beta \xrightarrow{T} A\beta = K\beta$ where $\beta = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

   $A = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}$

2. Determine the image of $\alpha = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ under the matrix transformation $\alpha \xrightarrow{T} A\alpha$ for each A, and describe the geometric effect of the transformation.

   (a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ SAME

   (b) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2a_1 \\ a_2 \end{bmatrix}$ slides terminal point two units on $X_1$ Axis.

3. Show that the following transformation is not linear.
   $(a,b) \xrightarrow{T} (a,2)$ or $T(a,b)=(a,2)$.

   $T\{K(a,b)\} = (Ka, 2)$

   $KT\{(a,b)\} = K(a,2) = (Ka, 2K)$

   $T\{(a,b)+(c,d)\} = T\{(a+c),(b+d)\} = (a+c, 2)$

   $T\{(a,b)\} + T\{(c,d)\} = (a,2)+(c,2) = (a+c, 4)$

# CHAPTER ELEVEN: CONVEX SETS.

The following are properties of linear inequalities:

If $a \geq b$ then $a+m \geq b+m$ ( or $a-m \geq b-m$); If $a \geq b$ and $m > 0$ $am \geq bm$, and $\frac{a}{m} \geq \frac{b}{m}$ ; If $a \geq b$ and $m < 0$, $am \leq bm$ and $\frac{a}{m} \leq \frac{b}{m}$.

An inequality which fails to be true for certain values of the variables involved is called a conditional inequality. The solution is the set of all values of that variable for which the inequality is true.

A set of points in n-space is said to be convex if for every pair of points $\alpha, \beta$, in the set, the line segment joining these points is also in the set. A set is considered to be convex if it contains fewer than two points. The intersection of two or more convex sets is convex. A linear function $f= k_1x_1+k_2x_2$ defined at every point of a convex polygon assumes its minimum and maximum values ( for the polygon) only at extreme points of the convex polygon.

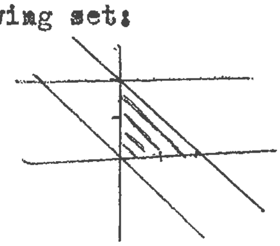
EXERCISES:

1. Solve for x: $6-3x \geq 12$ $\quad x \leq -2$
   $-3x \geq 6$

2. Find any superfluous inequalities in the following set:

$$\begin{cases} x_1 + x_2 < 2 \\ x_1 + x_2 < 0 \\ x_1 \geq 0 \\ x_2 \leq 2 \end{cases}$$

$x_1 + x_2 < 0$ superfluous



3. Solve for x: $\frac{1}{x} > \frac{2-x}{x}$

$\frac{x}{x^2} > \frac{2x-x^2}{x^2}$

$x > 2x - x^2$

$0 > (x-x^2)$

$0 > x(1-x)$

$x>0$ $\quad 1-x<0$ $\quad$ or $\quad x<0$ $\quad 1-x>0$

$x>0$ $\quad 1<x$ $\quad\quad x<0$ $\quad 1>x$

$1<x$ $\quad\quad x<0$

either $x<0$ or $x>1$

## CHAPTER TWELVE: LINEAR PROGRAMMING.

In general, a problem of linear programming is that of finding nonnegative values of a number of variables for which a certain linear function of these variables assumes the greatest ( or the leat ) possible value while subject to certain linear constraints. The linear programming problem can be expressed in matrix notation. Every linear programming problem has associated with it another linear programming problem caled called its dual, and the solution of one exists if and only if the other has a solution. The following chart is helpful in remembering how to write the dual of a linear problem:

| maximum problem | minimum problem |
|---|---|
| $1Z_1 + 3Z_2 \leq 4$<br>$1Z_1 + 1Z_2 \leq 2$<br>$4Z_1 + 5Z_2 = f$ | $\begin{array}{ccc} 1X_1 & 3X_1 & 4X_1 \\ + & + & + \\ 1X_2 & 1X_2 & 2X_2 \\ \geq & \geq & = \\ 4 & 5 & g \end{array}$ |

The simplex method will be split into three parts: (A) setting up the proper matrix, (B) performing the operations on the matrix, and (C) interpreting the final matrix. For part A, regardless of whether the original problem is a maximum problem or a minimum problem the form Maximize $f=R^tZ$, subject to $\begin{cases} BZ \leq C^T \\ Z \geq 0 \end{cases}$ can be arrived at. From this form creat a matrix: $\left[\begin{array}{c|c|c} B & I_m & C^T \\ \hline R^T & 0 & f \end{array}\right]$

Part B consists of the following steps on the matrix.

Step 1. Choose any column ( except the last ) whose last entry is positive. Step 2. Find a pivot entry by dividing each nonzero entry ( except the last ) of the chosen column into the corresponding entry of the last column, and selecting as the pivot, the entry which yields the smallest of the resulting nonnegative ratios. The row which includes the pivot is called the pivot row. The purpose here is to keep the entries of the last column nonnegative. Step 3. By adding multiples of the entries of the pivot row to those of the other rows, make all other entries of the chosen column zero. Step 4. Repeat the first three steps until all entries in the last row are nonpositive. Step 5. By using only the first two elementary row operations create a submatrix $I_m$ in part of the space formerly occupied by B $I_m$ . Part C upon proper interpretation of the resulting equivalent matrix will yield the solution to the maximum problem, $R^tZ$ and minimum CX. In the matrix obtained $-X^t$ occupies the position formerly occupied by the nullvector. To find Z examine each column to see if it has the number 1 and the rest 0's. if it does then the value of $Z_i$ for that column i is equal to the last entry in the same row with the 1. If it is not such a column the value of $Z_i$=0. To find f consider the entry in the lower right hand corner of the final matrix. Set it equal to zero and solve for f.

**EXERCISES:**

1. Two oil refineries produce three grades of gasoline A,B, and C. At each refinery, the various grades of gasoline are produced in a single operation so that they are in fixed proportions. Assume that one operation at Refinery 1 produces 1 unit of A, 3 units of B, and 5 units of C. Refinery 1 charges $300 for what is produced in one operation, and Refinery 2 charges $500 for the production of one operation. A consumer needs 100 units of A, 340 units of B, and 150 units of C. How should the orders be placed it the consumer is to meet his needs most economically?

| | 1 | 2 | UNITS NEEDED |
|---|---|---|---|
| A | 1 | 1 | 100 |
| B | 3 | 4 | 340 |
| C | 1 | 5 | 150 |
| COST PER OPERATION | $300 | $500 | |

$A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 1 & 5 \end{bmatrix}$ $R = \begin{bmatrix} 100 \\ 340 \\ 150 \end{bmatrix}$ $C = [300 \ . \ 500]$

$$\begin{cases} 1X_1 + 1X_2 \geq 100 \\ 3X_1 + 4X_2 \geq 340 \\ 1X_1 + 5X_2 \geq 150 \end{cases} \qquad \begin{cases} X_1 \geq 0 \\ X_2 \geq 0 \end{cases}$$

$\begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \geq \begin{bmatrix} 100 \\ 340 \\ 150 \end{bmatrix}$ $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \geq 0$ MINIMIZE $300X_1 + 500X_2$

OR $[300 \ \ 500] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ OR $CX$

Geometric Solution

$$\begin{array}{r} X_1 + X_2 = 100 \\ X_1 + 5X_2 = 150 \\ \hline -4X_2 = -50 \\ X_2 = 12\frac{1}{2} \\ \cancel{X_1} = 87\frac{1}{2} \end{array} \qquad \begin{array}{r} X_1 + X_2 = 100 \\ 3X_1 + 4X_2 = 340 \\ \hline -X_2 = -40 \\ X_2 = 40 \\ X_1 = 60 \end{array} \qquad \begin{array}{r} 3X_1 + 4X_2 = 340 \\ X_1 + 5X_2 = 150 \\ \hline -11X_2 = -110 \\ X_2 = 10 \\ X_1 = 100 \end{array}$$

$300(12\frac{1}{2}) + 500(87\frac{1}{2}) = 47500$

$300(60) + 500(40) = 38,000$

$300(100) + 500(10) = 35,000$

Minimum is to operate factory 1, 100 times and factory 2, 10 times.

2. Set up the dual of each of the following problems.

(a) Minimize $g = x_1 + x_2$ subject to $\begin{cases} X_1 + 2X_2 \geq 12 \\ X_1 + 3X_2 \geq 6 \end{cases}$ $\begin{cases} X_1 \geq 0 \\ X_2 \geq 0 \end{cases}$

__dual__ Maximize $f = 12z_1 + 6z_2$ subject to $\begin{cases} Z_1 + Z_2 \leq 1 \\ 2Z_1 + 3Z_2 \leq 1 \end{cases}$ $\begin{cases} Z_1 \geq 0 \\ Z_2 \geq 0 \end{cases}$

(b) Minimize $g = x_1 + 2x_2$, subject to $\begin{cases} 2X_1 + X_2 \geq 8 \\ X_1 + X_2 \geq 6 \\ X_1 + 2X_2 \geq 9 \end{cases}$ $\begin{cases} X_1 \geq 0 \\ X_2 \geq 0 \end{cases}$

__dual__ Maximize $f = 8z_1 + 6z_2 + 9z_3$ $\begin{cases} 2Z_1 + Z_2 + Z_3 \leq 8 \\ Z_1 + Z_2 + 2Z_3 \leq 6 \end{cases}$ $\begin{cases} Z_1 \geq 0 \\ Z_2 \geq 0 \end{cases}$

(c) Maximize $f = 2z_1 + z_2$ subject to $\begin{cases} Z_1 + Z_2 \geq 1 \\ Z_1 - Z_2 \geq -1 \\ Z_1 + 2Z_2 \leq 4 \end{cases}$ $\begin{cases} Z_1 \geq 0 \\ Z_2 \geq 0 \end{cases}$

__dual__ Minimize $g = -x_1 + x_2 + 4x_3$ subject to $\begin{cases} -X_1 - X_2 + X_3 \geq 2 \\ -X_1 + X_2 + 2X_3 \geq 1 \end{cases}$ $\begin{cases} X_1 \geq 0 \\ X_2 \geq 0 \\ X_3 \geq 0 \end{cases}$

3. Solve the following by the simplex method.

(a) Maximize $f=z_1+2z_2$ subject to $\begin{cases} z_1+z_2 \le 4 \\ z_1+4z_2 \le 7 \end{cases}$ $\begin{cases} z_1 \ge 0 \\ z_2 \ge 0 \end{cases}$

$$B=\begin{bmatrix}1&1\\1&4\end{bmatrix} \quad C^T=\begin{bmatrix}4\\7\end{bmatrix} \quad R^T=\begin{bmatrix}1&2\end{bmatrix} \quad Z=\begin{bmatrix}z_1\\z_2\end{bmatrix}$$

$$\left[\begin{array}{cc|cc|c}1&1&1&0&4\\1&4&0&1&7\\ \hline 1&2&0&0&f\end{array}\right] \underset{\substack{-R_1+R_2\\-R_1+R_3}}{\sim} \left[\begin{array}{cc|cc|c}1&1&1&0&4\\0&3^*&-1&1&3\\ \hline 0&1&-1&0&f-4\end{array}\right] \underset{\substack{-\frac{1}{3}R_2+R_1\\-\frac{1}{3}R_2+R_3}}{\sim} \left[\begin{array}{cc|cc|c}1&0&1\frac{1}{3}&-\frac{1}{3}&3\\0&3&-1&1&3\\ \hline 0&0&-\frac{2}{3}&-\frac{1}{3}&f-5\end{array}\right]$$

$$\underset{\frac{1}{3}R_2}{\sim} \left[\begin{array}{cc|cc|c}1&0&1\frac{1}{3}&-\frac{1}{3}&3\\0&1&-\frac{1}{3}&\frac{1}{3}&1\\ \hline 0&0&-\frac{2}{3}&-\frac{1}{3}&f-5\end{array}\right] \qquad -X^T=\begin{bmatrix}-\frac{2}{3}&-\frac{1}{3}\end{bmatrix} \quad X=\begin{bmatrix}\frac{2}{3}\\\frac{1}{3}\end{bmatrix} \quad Z=\begin{bmatrix}3\\1\end{bmatrix} \quad f=5$$

(b) Maximize $f=2z_1+z_2+6z_3+z_4$ subject to $\begin{cases} z_1+3z_2+z_3+z_4 \le 4 \\ z_1+z_3+2z_4 \le 5 \\ z_2+z_3 \le 2 \end{cases}$ $\begin{cases} z_1 \ge 0 \\ z_2 \ge 0 \\ z_3 \ge 0 \\ z_4 \ge 0 \end{cases}$

$$\left[\begin{array}{cccc|ccc|c}1^*&3&1&1&1&0&0&4\\1&0&1&2&0&1&0&5\\0&1&1&0&0&0&1&2\\ \hline 2&1&6&1&0&0&0&f\end{array}\right] \underset{\substack{-R_1+R_2\\-2R_1+R_4}}{\sim} \left[\begin{array}{cccc|ccc|c}1&3&1&1&1&0&0&4\\0&-3&0&-1&-1&1&0&1\\0&1&1^*&0&0&0&1&2\\ \hline 0&-5&4&-1&-2&0&0&f-8\end{array}\right]$$

$$\underset{\substack{-R_3+R_1\\-4R_3+R_4}}{\sim} \left[\begin{array}{cccc|ccc|c}1&2&0&1&1&0&0&2\\0&-3&0&-1&-1&1&0&1\\0&1&1&0&0&0&1&2\\ \hline 0&-9&0&-1&-2&0&-4&f-16\end{array}\right] \underset{R_1+R_2}{\sim} \left[\begin{array}{cccc|ccc|c}1&2&0&1&1&0&0&2\\0&-1&0&0&0&1&0&3\\0&1&1&0&0&0&1&2\\ \hline 0&-9&0&-1&-2&0&-4&f-16\end{array}\right]$$

$$+X^T=(-2\ \ 0\ \ -4) \qquad Z=\begin{bmatrix}2\\0\\2\\2\end{bmatrix} \qquad f=16$$

$$X=\begin{bmatrix}2\\0\\4\end{bmatrix}$$