A Brief Study of Topology

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A BRIEF STUDY OF TOPOLOGY

A Special Studies

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A BRIEF STUDY OF TOPOLOGY

Topology is the study of topological properties of figures--those properties which do not change under "elastic" motion. A figure may be theoretically stretched, twisted, bent, or cut and tied in a knot. The main restriction on such operations is that distinct points remain distinct; two points cannot be made to converge into one point.

Topology is generally divided into two branches: set topology and algebraic topology. This division is mostly a matter of convenience rather than of logic; however, there is much overlapping between the two branches. Set topology discusses the nature of a topological space, the properties of sets of points, the definitions of limits and continuity, the special properties of metric spaces, and questions concerning separation and connectedness. Algebraic topology deals with groups which are defined on a space, their structure and invariants. Its most important subdivisions are the theories of homology and homotopy.

One of the properties studied in this branch of mathematics is the group property. A group (which is not synonymous with a set) consists of a set and an operation performed on that set, such as the operation of addition performed on the set of real numbers. In order for such a combination to be called a group, the following criteria must be met:

1) the answer obtained when the operation is performed on members of the set must also be a member of the set.

2) the associative property must hold:
$$f(gh) = (fg)h$$

3) the set must contain an identity element

4) the set must contain the inverse of each element.

The term "abelian" is used to describe a group if it is commutative. Groups which are definitely not commutative are termed "nonabelian."

A <u>subgroup</u> of a group is a subset (of the original set) which is closed under multiplication and inversion.

Another property studied is that of a <u>morphism</u> (or homomorphism). A function is called a morphism of it preserves products, that is if $f(g_1g_2) = f(g_1)f(g_2)$. A morphism is <u>epic</u> if the function is onto, which means that the image of the function is equal to the range of the function. A morphism is <u>monic</u> if it is a one-to-one correspondence function. An <u>isomorphism</u> is a morphism which is both epic and monic.

The properties of metric spaces are also important to the study of topology. A metric space consists of a set, M, together with a distance function, d, such that $d: M \times M \rightarrow R$ with all x, y, and z members of M. The restrictions on metric spaces are:

1) $d(x,y) \geq 0$
2) $d(x,y) = 0$ if and only if $x = y$
3) $d(x,y) = d(y,x)$
4) $d(x,z) \leq d(x,y) + d(y,z)$

A topological space is said to be connected if, and only if, it is not the union of two nonempty disjoint sets. An equivalent definition is as follows: A topological space X is connected if, and only if, X and the null set are the only subsets of X which are both open and closed. Two points in X are said to be connected if, and only if, there is a connected subset of X which contains both of them. If a space is not connected, it separates into a number of components each of which is connected and is not contained in any other connected subset. A space X is said to be locally connected at a point x of X if every open set U containing x contains an open, connected set V of which x is an element. If X is a locally connected Gausdorff space, then all components of open sets are open. X is called arcwise connected if every two points of X can be joined by an arc lying in X. If X is a metric space which is locally compact, connected, and locally connected, then it is arcwise connected.

Although topology is now considered to be a branch of pure mathematics, its early development came in connection with problems in physics. Gustav Robert Kirchhoff (1847) developed the theory of linear graphs in connection with his study of electrical networks. James Clerk Maxwell (1873) made contributions to the theory of connectivity in his work on electromagnetic field theory. Henri Poincare (1895) laid the foundations of algebraic topology while working on problems in celestial mechanics. In the mid-twentieth century, topological ideas are of great importance in the development of modern function theory, in theories of nonlinear differential equations, in differential geometry, and in dynamics. Topology is thus an indispensable tool of the modern mathematician, pure or applied.

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Chapter 1 Source: McCarty, George. Topology. St. Louis: McGraw-Hill, 1967.

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A) $T_1 = \{y : |y-1| < 1 \text{ and } |y+1| > 1\} = \{0 < y < 2\}$

$T_2 = \{y : |y-1| < 2 \text{ and } |y+1| > \frac{1}{2}\} = \{-\frac{1}{2} < y < 3\}$

$T_3 = \{y : |y-1| < 3 \text{ and } |y+1| > \frac{1}{3}\} = \left\{\begin{matrix} -2 < y < -\frac{4}{3} \\ -\frac{2}{3} < y < 4 \end{matrix}\right\}$

$T_4 = \{y : |y-1| < 4 \text{ and } |y+1| > \frac{1}{4}\} = \left\{\begin{matrix} -3 < y < -\frac{5}{4} \\ -\frac{3}{4} < y < 5 \end{matrix}\right\}$

etc.

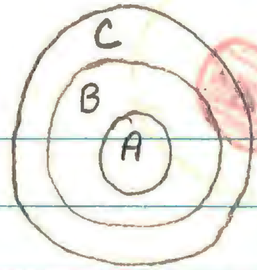
$\therefore \cup \mathcal{T} = \{y : y \neq -1\}$

$\cap \mathcal{T} = \{0 < y < 2\}$

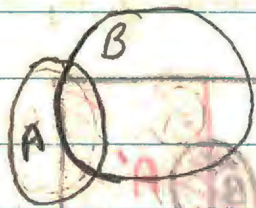
B) i) $\emptyset$, the empty set, contains no elements. By definition, the complement of a set which contains no elements would be the set which contains all elements. The set which contains all elements is U, the universe.

Conversely, the complement of the set which contains all elements would be the set which contains no elements, the empty set, $\emptyset$.

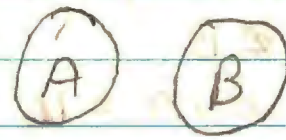
ii) Let $A = \{1, 2, 3\}$. By definition, a subset of this set must not contain any elements which are not contained in set $A = \{1, 2, 3\}$. Therefore, A, which contains 1, 2, and 3 fulfills these requirements, and $A \subset A$.

iii) 

If A is contained in B, and B is contained in C, then A contained in C is implied.

iv) 

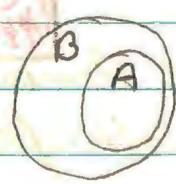
OR

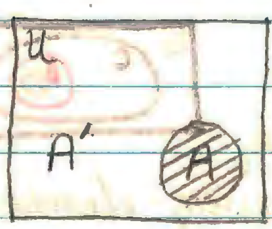


If $A \cap B \neq A$, then one of the two diagrams above illustrates the situation. In either case, some element of A is not contained in B, and $A \not\subset B$.

If $A \cup B \neq B$, the diagrams also illustrate the situation; some element of A is not in B, and $A \not\subset B$.

If $A \cap B = A$ and $A \cup B = B$, then $A \subset B$.



v) 

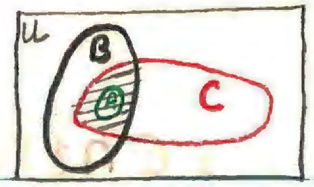
No element in A is in A'. So

$$A \cap A' = \emptyset$$

The elements in either set combined make up the set $\mathcal{U}$. So

$$A \cup A' = \mathcal{U}$$

viii) If $A \subset B$ and $A \subset C$,

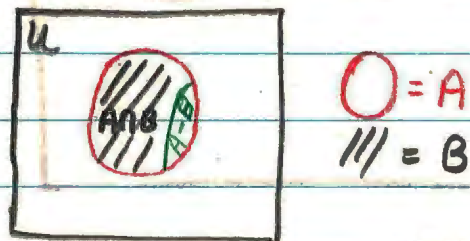


Then all elements of set A are also elements of sets B and C, and $A \subset (B \cap C)$

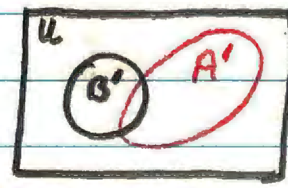
ix) Let $A = \{1,2,3,4,5\}$ Let $B = \{1,2,3\}$

Then $(A-B) = \{4,5\}$ and $(A \cap B) = \{1,2,3\}$

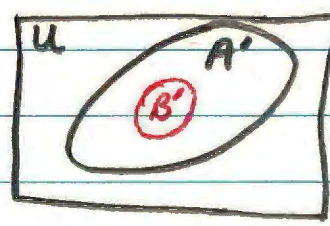
$(A-B) \cup (A \cap B) = \{1,2,3,4,5\} = A$



x) If $B' \not\subset A'$, then there exists some element x which is an element of B' and not an element of A'. This means $x \in A$ and $x \notin B$, and $A \not\subset B$.



If $B' \subset A'$, then every element which is not in B, is not in A, and $A \subset B$.



c) Prove: $T \cup \bigcap \mathcal{S} = \bigcap \{T \cup S : S \in \mathcal{S}\}$

Let $x \in T \cup \bigcap \mathcal{S}$; then $x \in T$ or $x \in \bigcap \mathcal{S}$. If $x \in T$, then $x \in T \cup S$ and $x \in \bigcap \{T \cup S : S \in \mathcal{S}\}$. Therefore

$$T \cup \bigcap \mathcal{S} \subset \bigcap \{T \cup S : S \in \mathcal{S}\}$$

If $x \in \bigcap \mathcal{S}$, there exists some $S \in \mathcal{S}$ with $x \in S$, and

Chapter 2

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A)

$f \backslash g$	1	f_a	f_b	f_c	r	r^2
1	1	f_a	f_b	f_c	r	r^2
f_a	f_a	1	r	r^2	f_b	f_c
f_b	f_b	r^2	1	r	f_c	f_a
f_c	f_c	r	r^2	1	f_a	f_b
r	r	f_b	f_c	f_a	r^2	1
r^2	r^2	f_c	f_a	f_b	1	r

B) i) yes

ii) no ; there is no identity element, no group property

iii) yes

iv) yes

v) no ; $f(gh) \neq (fg)h$

vi) no ; no identity element

C) assume

$$(a_1 a_2)(a_3 \dots a_K) = a_1(a_2 a_3 \dots a_K)$$

or any other arrangements of parentheses

then

$$[(a_1 a_2)(a_3 \dots a_K)](a_{K+1}) = [a_1(a_2 a_3 \dots a_K)](a_{K+1})$$

D) Let $hg = g$; then

$$e = gg^{-1} = (hg)g^{-1} = h(gg^{-1}) = he = h$$

Let $hg = e$; then

$$h = he = h(gg^{-1}) = (hg)g^{-1} = eg^{-1} = g^{-1}$$

SOURCE: FIRST CONCEPTS OF TOPOLOGY by Chinn

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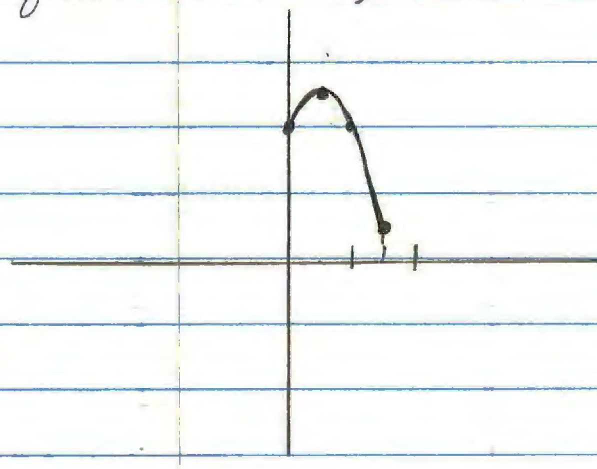
1) minimum and maximum of $f(x) = 4 + 2x - x^2$, $0 \leq x \leq 3$

$f'(x) = 2 - 2x = 0$

$2x = 2$

maximum $x = 1$, $f(x) = 5$



minimum $f(x) = 1$ $x = 3$

no values of x in the interval for values of $y > 5$ or $y < 1$

only one value of x when $1 \leq y < 4$ or $y = 5$

two values of x when $4 \leq y < 5$

2) when $x = \sqrt[3]{5}$, $f(x) = 0$

because 0 lies between -4 and 3, $\sqrt[3]{5}$ must lie between 1 and 2

3) when $x = 1$, $x^2 - 2x - 4 = -5 = y$

$x = 2$, $y = -4$

$x = 3$, $y = -1$

$x = 4$, $y = 4$

$\therefore$ $y = 0$ when $3 < x < 4$

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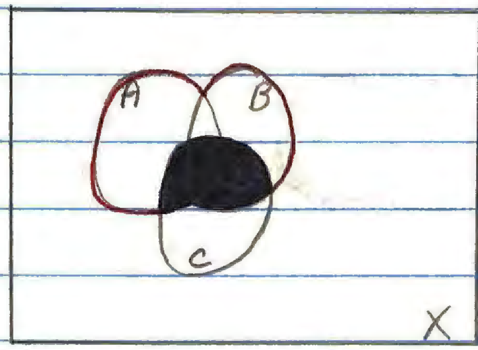
1)



$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

Let $x \in (A \cup B) \cap C$. Then either $x \in A$ and $x \in C$ or $x \in B$ and $x \in C$.

If $x \in A$ and $x \in C$, then $x \in (A \cap C)$ and $x \in (A \cap C) \cup (B \cap C)$.

If $x \in B$ and $x \in C$, then $x \in (B \cap C)$ and $x \in (A \cap C) \cup (B \cap C)$.

$\therefore\ (A \cup B) \cap C \subset (A \cap C) \cup (B \cap C)$

Let $x \in (A \cap C) \cup (B \cap C)$. Then either $x \in (A \cap C)$, which means $x \in A$ and $x \in C$, or $x \in (B \cap C)$, which means $x \in B$ and $x \in C$.

If $x \in A$ and $x \in C$, then $x \in (A \cup B)$ and $x \in (A \cup B) \cap C$.

If $x \in B$, then $x \in (A \cup B)$ and if also $x \in C$, then $x \in (A \cup B) \cap C$.

$\therefore\ (A \cap C) \cup (B \cap C) \subset (A \cup B) \cap C$

$$\therefore\ (A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

4) (a) Let $y \in f(A \cup B)$ then $x \in A \cup B$ when $fx = y$.
If $x \in A$, then $y \in fA$. If $x \in B$, then $y = fB$. In either case $y \in fA \cup fB$. $\therefore f(A \cup B) \subset fA \cup fB$

Let $y \in fA \cup fB$ then $y \in fA$ or $y \in fB$.
If $y \in fA$, then $x \in A$. If $y \in fB$, $x \in B$. In either case $x \in A \cup B$ so $y \in f(A \cup B)$. $\therefore fA \cup fB \subset f(A \cup B)$

$$\therefore f(A \cup B) = fA \cup fB$$

(b) Let $y \in f(A \cap B)$. Then $x \in A \cap B$ when $fx = y$, and $x \in A$ and $x \in B$. This means $y \in fA$ and $y \in fB$, so $y \in fA \cap fB$.

$$\therefore f(A \cap B) \subset fA \cap fB$$

(c) Let $x \in A$; if $A \subset B$, then also $x \in B$.
Then if $fx = y$ $y \in fA$ and $y \in fB$

$$\therefore fA \subset fB$$

5. a) a circle with S as center
b) a line through S
c) an arc of a circle
d) twice as large
e) semicircle between S and N, excluding N

Let $y \in f^{-1}(A \cap B)$, then $x \in A \cap B$ and $x \in A$ and $x \in B$. This means $y \in f^{-1}A$ and $y \in f^{-1}B$ so that
$y \in f^{-1}A \cap f^{-1}B$.
$\therefore f^{-1}(A \cap B) \subset f^{-1}A \cap f^{-1}B$

Let $y \in f^{-1}A \cap f^{-1}B$, then $y \in f^{-1}A$ and $y \in f^{-1}B$. This means $x \in A$ and $x \in B$ so that $x \in A \cap B$ and $y \in f^{-1}(A \cap B)$.
$\therefore f^{-1}A \cap f^{-1}B \subset f^{-1}(A \cap B)$

$\therefore f^{-1}(A \cap B) = f^{-1}A \cap f^{-1}B$

(b) $f^{-1}Y = X$ $\quad f^{-1}\phi = \phi$

(c) $f^{-1}A \subset f^{-1}B$

8) $f : X \to Y \quad g : Y \to Z$

Let $x \in (gf)^{-1}C$, then $gfx \in C$ and $fx \in g^{-1}C$ which means $x \in f^{-1}(g^{-1}C)$. $\therefore (gf)^{-1}C = f^{-1}(g^{-1}C)$.

If $x \neq x'$, $fx \neq fx'$ and $gfx \neq gfx'$. Therefore gf is 1-1.

By definition $gy = z$; $fx = y$, $gfx = z$, $(gf)^{-1}z = x$
$y = (g^{-1})z$, $x = (f^{-1})y$
$x = (f^{-1})(g^{-1})z = (gf)^{-1}z$

Intuitive Concepts in Elementary Topology
by B.H. Arnold

Chapter 0, Part 1

I.

	Is it a statement?	Is it True?
a)	yes	yes
b)	no	
c)	no	
d)	no	
e)	no	
f)	no	
g)	yes	no
h)	yes	no
i)	yes	no
j)	yes	yes
k)	yes	~~yes~~ no
l)	yes	no
m)	no	
n)	yes	no
o)	yes	no
p)	yes	no
q)	yes	no
r)	yes	yes
s)	yes	yes
t)	yes	no
u)	yes	no
v)	yes	yes

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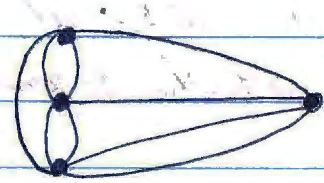
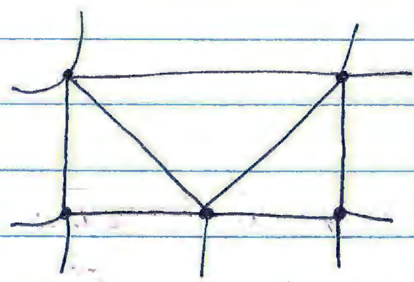
1. a) no ; all vertices are odd
b) yes ; yes



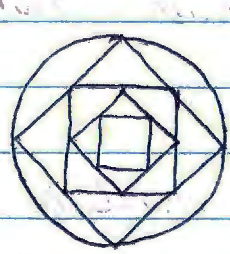
c) networks a, b, c, e, g, and h

2.

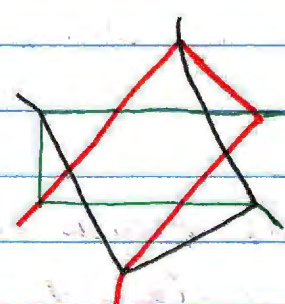


no ; because there are 3 odd vertices

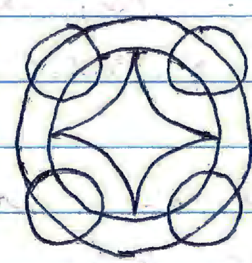
4.



(a)

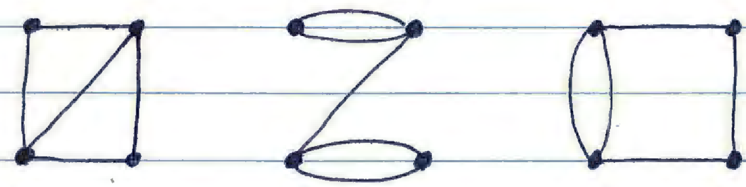


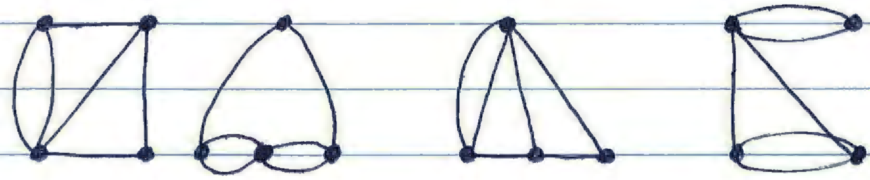
(b)



(c)

one path 3 paths one path

7) b) 

c) 

8) a) e

b) none

10) a) where more than 2 arcs meet

b) > 2 ; odd