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A History of Mathematics Through the Time of Greek Geometry

Janet Moffett

Ouachita Baptist University

A HISTORY OF MATHEMATICS THROUGH THE TIME OF GREEK GEOMETRY

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Janet Moffett

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The concept of numbers and the process of counting developed long before the time of recorded history. The manner of its development is not known for certain but is largely conjectual. It is presumed that man, even in most primitive times, had some number sense, at least to the extent of recognizing "more" or "less" when objects were added or taken away from a small group. As civilization progressed it became mecessary for man to count. He needed to know the number of sheep he owned, the number of people in his tribe, etc. The most logical method was to allow some object to stand for each thing being counted. Since the fingers were so convenient, they were most often used as such a means. The plan of indicating numbers by the digits of one or both hands is so natural that it was almost universal amoung early races. The number five was generally represented by the open hand and it is said that in almost all languages the words five and hand are derived from the same root. It is possible that in early times man did not count beyond five and anything larger was represented by multiples of it. It is possible that the Roman symbol X, for ten, represents two V's. Most races, however, apparently used both hands and could count up to ten. Some tribes seem to have gone further and, by using their toes, could count up to twenty. Counts could also be made by making collections of pebbles or sticks, by making scratches in the dirt or on a stone, by cutting notches in a piece of wood, or by tying knots in a string.

The first efforts at graphical representation of numbers came long after people learned to count. The earliest method of keeping count was probably that of some tally system involving some physical objects such as pebbles or sticks. The earliest numerals were notches in a stick or marks on a piece of pottery. This served the purpose needed by primitive man who, because he led a simple life, had no need for a more complex number system. As civilizations developed more, men devised a way of representing numbers by means of pebbles or counters arranged in sets of ten. This developed into the abacus or swan-pan. Vocal sounds were probably used to designate a number of objects in a small group long before there were separate symbols for the numbers. Sounds differed according to the kind of objects being counted. The abstract idea of two, signified by a particular sound independant of the object counted, was a long time in developing.

As life grew more complex, it became necessary for man to count larger numbers of objects. Numbers larger than ten were a problem and a variety of methods were devised for the higher numbers. Since man found his fingers useful in counting, a system with a base of ten naturally evolved. Although probably most widely used, bases other than that of ten also evolved. The quinary scale, having a base of five, was also widely used. There is also evidence that twelve may have been used as a base in prehistoric times. This is based on the common use of twelve in many measurements of today. The sexagesimal scale, or number system based on sixty, was used by the ancient Babylonians, and is still used when measuring time and angles in minutes and seconds.

Perhaps the earliest type of written number system that was developed is that of a simple grouping system. In such a system, some number b is selected for a base and symbols are adopted for 1, b, b^2, b^3, and so on. Any number may then be expressed by using these symbols additively, each symbol being repeated the required number of times. An example of a simple grouping system is that of Egyptian hieroglyphics which was employed as far back as 3400 B.C. It was based on the scale of ten and represented by the following symbols:

1	𓏺	a vertical staff
10	𓎆	a heel bone
10^2	𓍢	a scroll
10^3	𓆼	a lotus flower
10^4	𓂭	a pointing finger
10^5	𓆐	a burbot fish
10^6	𓁨	a man in astonishment

Numbers were expressed in the following manner:

13015 = 1(10) + 3(10) + 1(10) + 5 = 𓏺𓏺𓏺𓏺𓏺𓎆𓆼𓆼𓆼𓂭

The early Babylonians used cuniform (wedge-shaped) characters as early as 2000 B.C. Their system used a base of ten for numbers less than 60 and was often simplified by using a subtraction symbol.

1 = Y

10 = <

subtractive symbol =

25 = 2(10) + 5 = << YYYYY

38 = 4(10) - 2 = <<<< YY

Numbers larger than sixty were based on the sexagesimal system and employed the principle of position.

$524,551 = 2(60^3) + 25(60^2) + 42(60) + 31 =$ 

The Greek numerical system was a ciphered numerical system utilizing the Greek alphabet together with symbols for digamma, koppa, and sampi which have become obsolete. The system uses a base ten with the letters representing 1-10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 200, 300, 400, 500, 600, 700, 800, 900. Accompanying bars or accents were used for larger numbers.

Our present number system, Hindu-Arabic, is named after the Hindus who are believed to have invented it and the Arabs who transmitted it to western Europe. Several different claims have been made with respect to the origin of the system. These claims included the Arabs, Persians, Egyptians and Hindus. The true origin of the present day numerials may have been a conglomeration from different sources. India is the first country known to have used most of the present numeral forms. The earliest preserved examples were found on some stone columns erected in India about 250 B.C. Near the close of the eighth century, some astronomical tables in India are believed to have been translated into Arabic and the numerals became known to Arabian scholars.

Early mathematics originated primarily as a practical science to assist in agriculture and engineering pursuits. These pursuits required the computation of a usable calendar, the development of a system of weights and measures, the creation of surveying methods for canal and reservoir construction, and the evolution of financial and commercial practices for raising and collecting taxes and for purposes of trade. Geometry is believed to have had its origin in land-surveying. Some form of survey land was practiced from very early times, but it is asserted that the origin of geometry was in Egypt. The periodical flooding of the Nile swept away the landmarks in the valley of the river. By altering its course it increased or decreased the taxable value of adjoining lands and therefore rendered a tolerably accurate system of surveying indispensable to the Egyptians. This led to a systematic study of the subject by the priests.

The Egyptians were very particular about the exact orientation of their temples. They, therefore, had to obtain with accuracy a north south line, and also an east west line. They first obtained a north south line by observing the points on the horizon where a star rose and set, and taking a plane midway between them, they could obtain a north south line. In order to obtain an east west line, which had to be drawn at right angles to the line previously obtained, professional "rope-fasteners" were employed. These men used a rope ABCD divided by knots or marks at B and C, so that the lengths AB, BC, and CD were in the ratio 3:4:5. The length BC was placed along the north south line, and pegs inserted at the knots B and C. The piece BA was then rotated around peg B, while peice CD was rotated around peg C, until the ends A and D met. Peg A was then inserted at that point forming a triangle ABC whose sides were in the ratio 3:4:5. The angle to the triangle at B would then be a right and with AB giving an east west line. A similar method is used today by practical engineers for measuring a right angle.

All the specimens of Egyptian geometry known deal only with particular numerical problems and not general theorems. Even if a result were stated as universally true, it was probably proved only by wide induction. The word geometry is derived form the Greek words γῆ, the earth, and μετρέω, I measure. This meaning is obviously based on the Egyptian practices since the Greek geometricians dealt with the science as an abstract one. They sought theorems which would be absolutely true.

The founder of the earliest Greek school of mathematics and philosophy was Thales. He was born about 640 B.C. at Miletus, and died there about 550 B.C. During the early part of his life he was engaged in commerce and public affairs. Thales first went to Egypt as a merchant and while there he studied astronomy and geometry. When he returned to Miletus, he abandoned business and public life and devoted himself to a study of philosophy and science. Thales geometrical works consisted of a number of isolated propositions not arranged in any logical sequence. He is credited with the following geometrical results:

1. A circle is bisected by any diameter.
2. The base angles of an isosceles triangle are equal.
3. The vertical angles formed by two intersecting lines are equal.
4. Two triangles are congruent if they have two angles and one side in each respectively equal.
5. An angle inscribed in a semicircle is a right angle.

The value of these results is in that the proof were deductive. This deductive character is his chief contribution to geometry.

Although it was Thales who first directed the general attention to geometry, it was Pythagoras who "changed the study of geometry into the form of a liberal education, for he examined its principles to the bottom and investigated its theorems in an .... intellectual manner."[1] Pythagoras was born at Samos about 569 B.C. and died in 500 B.C. He studied under Anaximander who was a student of Thales. He then studied in Egypt at Thebes or Memphis. Upon moving to Croton, a Dorian colony in the south of Italy, he opened schools which were crowded with enthusiastic audiences. He divided those who attended his lectures into two classes; probationers and Pythagoreans. Only to the Pythagoreans did he reveal his chief discoveries. The Pythagoreans formed a brotherhood with all things held in common, holding the same philosophical and political beliefs, engaged in the same pursuits, and bound by oath not to reveal the teaching or secrets of the school. The strict dicipline and secret organization gave the society a temporary supremacy in the state and with it the hatred of various classes. Finally, incouraged by his political opponents, the mob murdered Pythagoras and many of his followers. After the death of Pythagoras, the school re-established themselves as a philosophical and mathematical society with Tarentum as their headquarters. As such they continued to flourish for more than a hundred years.

Pythagoras himself did not publish any books. The assumption of his school was that all their knowledge was held in common and kept from the outside world. They also held that any fresh

[1] W. W. Rouse Ball, A Short Account of the History of Mathematics (MacMillan and Co., Limited: London, 1924), p. 19.

discovery must be referred back to their founder. It is therefore impossible to separate precisely the discoveries of Pythagoras himself from those of his school at a later date. Gradually, as the society became more scattered, the custom of keeping their knowledge from the outside world was abandoned and treatises containing the substance of their teaching and doctrines were written. The first book was composed by Philolaus about 370 B.C.

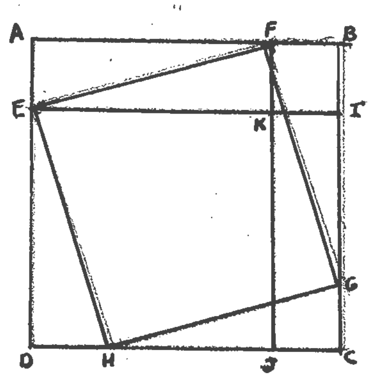
Pythagoras was the first to arrange the leading propositions of geometry in a logical order. The Pythagoreans divided the mathematical subjects with which they dealt into four divisions: numbers absolute or arithmetic, numbers applied or music, magnitudes at rest or geometry, and magnitudes in motion or astronomy. It is presumed that Pythagoras probably knew and taught the substance of what is contained in the first two books of Euclid about parallels, triangles, and parallelograms, and was acquainted with a few other isolated theorems. It is suspected, however, that many of his proofs were not rigorous, and the converse of a theorm was sometimes assumed without proof. Until the time of Pythagoras the only known numbers were rational numbers. Pythagoras showed that there were points on the number line not corresponding to any rational number. He did this by taking the length of the diagonal of a unit square. New numbers had to be invented to represent such points. The discovery of irrational numbers is one of the great milestones in the history of mathematics. Pythagorean teaching also included the following propositions:

1. It defined a point as unity having position.

2. The sum of the angles of a triangle was shown to be equal

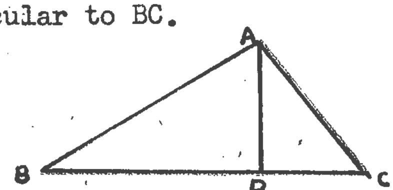
to two right angles.

3. Any square ABCD can be split up into two squares, FBIK and EKJD, and two equal rectangles, AFKE and KICJ: or it is equal to the square FBKI, the square EKOJ, and four times the triangle AEF.



If points are taken, G on BC, H on CD, and E on DA, so that BG, CH, and DE are each equal to AF, EFGH is a square, and the triangles AEF, BFG, CGH, and DHE are equal. Thus the square ABCD is also equal to the square EFGH and four times the triangle AEF. Therefore, the square EFGH is equal to the sum of FBIK and EKJD; or the square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.

4. Let ABC be a right-angled triangle, A being the right angle. Draw AD perpendicular to BC.



Triangles ABC and DBA are similar

$\therefore$ BC : AB = AB : BD

BC : AC = AC : DC

$\therefore AB^2 + AC^2 = BC(BD + DC) = BC^2$

5. Pythagoras showed that the plane about a point could be completely filled by equilateral triangles, by squares or by regular hexagons.
6. The Pythagoreans are believed to have attempted the quadrature of a circle. They stated that the circle was the most perfect of all plane figures.
7. They knew that there were five regular solids inscribable in a sphere.

About a century after the murder of Pythagoras, Archytas became the recognized head of the school. Archytas was one of the first to solve the problem of finding the side of a cube whose volume is double that of a given cube. He also introduced various mechanical devices for constructing curves and solving problems. His findings were not accepted because of the tradition of using only a compass and straightedge in geometrical proofs.

There were other schools at the time of the Pythagoreans. In the School of Chios the most outstanding member was Oenopides, who was born about 500 B.C. and died about 430 B.C. He is credited with the solution of the problem of constructing a perpendicular from a point to a line and with the construction of an angle at a given point equal to a given angle.

By the end of the fifth century B.C., the more prominent discoveries of the Pythagoreans became known to the outside world and the center of intellectual activity was transferred to Athens. The history of the Athenian school begins about 420 B.C. with the teaching of Hippocrates. The school was established on a permanent basis by Plato and Eudoxus. Eudoxus is believed to be the founder of the school in neighboring Cyzicus. The close intercourse between the schools kept their treatment of the subject identical. The schools were expecially concerned with three problems:

1. The duplication of a cube.
2. The trisection of an angle.
3. The squaring of a circle.

The three problems are insoluable with Euclidean geomentry. They were therefore destined to fail, but in their attempts they discovered many new theorms and processes. The school also collected all the geometrical theorems then known and arranged them systematically. The collections later comprised the major portion of the propositions in Euclid's Elements, Books I - IX, XI, and XII.

One of the greatest of the Greek geometricians was Hippocrates of Cos. He was born in Chios around 470 B.C. and came to Athens about 430 B.C. He wrote the first elementary text-book of geometry on which Euclid's Elements was probably founded. In his textbook, he introduced the method of "reducing" one theorem to another, which having already been proved, the thing proposed was true. He also elaborated the geometry of a circle: proving, among other propositions, that similar segments of a circle contain equal angles; and the angles subtended by a cord of a circle is greater than, equal to, or less than a right angle as the segment of the circle containing it is less than, equal to,,or greater than a semicircle.

Another member of the Athenian school was Plato. He was more of a philosopher than a mathematician, but exerted a great influence on his contemporaries and sucessors. He believed that the secret of the universe was to be found in number and form. He objected to the use of any instruments other than rulers and compasses in the construction of curves. This was accepted as a law to be observed in such problems. It is believed to be Plato who compiled series of definitions, postulates, and axioms. He also systematized the methods which could be used in attacking mathematical questions, and in particular directed attention to the value of analysis. He developed the method of analytical proof which begins by assuming that the theorem or problem is solved, then deducing some result. If the result is true and the steps reversible the original theorem is true.

Athens, together with the neighboring school of Cyzicus, continued to extend on the lines laid down by Hippocrates, Plato, and Eudoxus until the foundation of the university at Alexandria about 300 B.C. The Alexandrian school produced, within the first century of its existance three of the greatest mathematicians of antiquity - Euclid, Archimedes, and Apollonius. These three laid down the lines on which mathematics subsequently developed. They were the first to treat mathematics as a subject separate and distinct from philosophy. The department of mathematics at the Alexandrian school was placed under Euclid.

Euclid was of Greek decent and was born about 330 B.C. and died about 275 B.C. He is believed to have been educated at Athens. Euclid was the author of several works the most outstanding of which is his Elements. This treatise contains a systematic exposition of the leading propositions of elementary geometry and of the theory of numbers. The Greeks adopted it at once as the standard text-book on the elements of pure mathematics. The geometrical part of Elements is mostly a compilation from the works of previous writers. Books I and II are probably due to Pythagoras, Book III to Hippocrates, Book V to Eudoxus, and most of Books IV, VI, XI, and XII to the later Pythagorean or Athenian schools. The rest of the books are considered to be mainly original. Book I begins with the necessary preliminary definitions, postulates, and axioms. The 48 propositions of Book I fall into three groups. The first 26 deal mainly with properties of triangles and include the three congruence theorems. Propositions I 27 through I 32 establish the theory of parallel lines and prove that the sum of the angles of a triangle is equal to two right angles. The remainimg propositions of the book deal with parallelograms, triangles, and squares, with special reference to area relations. Proposition I 47 is the Pythagorean theorem with the final proposition, I 48, the converse of the Pythagorean theorem. Book II deals with the transformation of areas and the geometric algebra of the Pythagorean school. At the end of the book are two propositions which establish the generalization of the Pythagorean theorem known today as the "law of cosines." Book III contains theorems about circles, cords, tangents, and measurement of associated angles. In Book IV are found discussions of Pythagorean constructions, with straightedge and compasses, of regular polygons of three, four, five, six, and fifteen sides. Book V is an exposition of Eudoxus' theory of propertion. Book VI applies the Eudoxian theory of proportion to plane geometry. Books VII, VIII, and IX, contain a total of 102 propositions dealing with elemtary number theory. Book X deals with irrationals and is considered the most remarkable book by many scholars. The last three books XI, XII, and XIII deal with solid geometry with the exception of spheres.

Another student of the Alexandrian school was Archimedes. He lived from 287 B.C. to 212 B.C. After completing his studies at Alexandria, he returned to his home in Sicily. Rather than writing a systematic treatise like Euclid which could be understood by all students, Archimedes wrote a number of brilliant essays addressed chiefly to the most educated mathematicians of the day. He is best known today for his treatment of the mechanics of solids and fluids. His contemporaries esteemed his geometrical discoveries of the quadrature of a parabolic area and of a spherical surface, and his rule for finding the volume of a sphere. He also inaugurated the classical method of computing. .

The third great mathematician of this century was Apollonius of Perga, who produced a systematic treatise on the conic sections which not only included all that was previously known about them, but greatly extended the knowledge of these curves. Apollonius was born about 260 B.C. and died about 200 B.C. He studied in Alexandria and is thought to have lectured there. He spent some years at Pergamun in Pamphylia where a new university had been established after the fashion of Alexandria. He later returned to Alexandria and lived there until his death. Apollonius work contained around four hundred propositions and was divided into eight books. He so thoroughly investigated the properties of conic sections that little was left for his sucessors to add. The first four books deal with the elements of the subject with the first three founded on Euclid's work. In the fifth book he develops the theory of maxima and minima, applies it to find the center of curvature at any point of a conic and the evolute of the curve, and discusses the number of normals which can be drawn from a point to a conic. The sixth book relates propositions of similar conics. The seventh and eighth books are devoted to a discussion of conjugate diameters.

The mere art of calculation was taught to boys when quite young, it was stigmatized by Plato as childish, and never received much attention from Greek mathematicians. The third century B.C. is the most brilliant era in the history of Greek mathematics. It opened with the career of Euclid and closed with the death of Apollonius. The great mathematicians, however, were all geometricians and under their influence attention was directed solely to that particular branch of mathematics. Their works were so complete that it was 1800 years before Descartes opened the way to any futher progress in geometry.

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