1974

Intuitive Concepts in Elementary Topology

Gary Rothwell
Ouachita Baptist University

Intuitive Concepts in Elementary Topology

Presented to:

Dr. Don Carnahan

By:

Gary Rothwell

December 10, 1974

My hour special study in intuitive topology originated in a curiosity of what exactly topology was and how it might be related to physics, my field of interest. The book I used was, Intuitive Concepts in Elementary Topology, by B.H. Arnold. This book is designed as a sophomore-junior level three hour course. Needless to say, I didn't quite cover the whole book in an hour a week. I mainly stuck to the intuitive concepts. Intuitive topology is dealing with more physical objects where the point set topology involves set theory; their unions, intersections and subsets.

The first chapter was about statements and proofs in mathematics. A statement is any collection of symbols which forms a meaningful assertion and which has the property that this assertion is either definitely true or definitely false.

Examples

Statement - George Washington was a traitor.
No statement - Stop, thief!

With these statements we then went into truth tables explaining statements like; if p then q, p implies q, which are called implications. The last part was types of mathematical proofs. These include four main types:

1. Direct proof - Start with hypothesis that first statement is true and by valid steps deduct second is true.

Example

Prove that if n is an odd integer, then n^2 is an odd integer.

Proof: We have, by hypothesis, that n is an odd integer, then n^2 is an odd integer. Thus $\frac{1}{2}(n - 1)$ is an integer, say

$$\tfrac{1}{2}(n - 1) = m$$

Solving this equation for n gives

$$n = 2m + 1$$

and squaring both sides of the equation gives

$$n^2 = (2m + 1)^2 = 2(2m^2 + 2m) + 1.$$

But this last form shows that n^2 is an odd integer and the proof is complete.

2. Indirect proof - Proofs by contradiction - a direct proof for any one of the implications.

Example

Proof: Prove that $\sqrt{2}$ is irrational.
Proof by contradiction: Suppose the $\sqrt{2}$ is rational; then it can be expressed as a fraction in lowest terms, say

$$n/m = \sqrt{2}$$

where n and m are integers which nave no common factor except one. Then $n = \sqrt{2}\,m$ or, if we square both sides of this equation, $n^2 = 2m^2$. Thus n is an integer whose square is even and, n is even (opposite of first). Setting $n = 2r$ and subsituting in the above equation, we find

$$2r^2 = m^2$$

But this shows that m is an integer whose square is even; hence m is even, which contradicts the statement that n and m have no common factors except one.

3. Mathematical induction - First prove the statement true for T_1, then T_k implies T_{k+1}, or implication If T_1 and T_2 and T_3 and... T_{k-1} then T_k.

Example

Prove:
$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Prove for 1

$$\frac{1}{1(1+1)} = \frac{1}{1+1}$$

$$\frac{1}{1(2)} = \frac{1}{2}$$

$$\frac{1}{2} = \frac{1}{2}$$

If true for n then $n+1$ is true

$$\frac{n}{n+1} + \frac{1}{(n+1)(n+1+1)} = \frac{n+1}{(n+1)+1}$$

$$\frac{n}{(n+1)} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}$$

$$\frac{1}{n+1}\left[n + \left(\frac{1}{n+2}\right)\right] =$$

$$\frac{1}{n+1}\left(\frac{n^2+2n+1}{n+2}\right) =$$

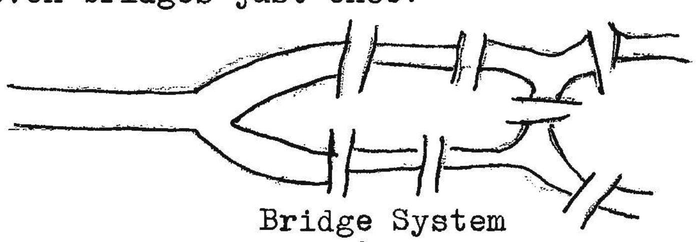
$$\frac{(n+1)\ (n+1)}{(n+1)\ (n+2)} =$$

$$\frac{n+1}{n+2} = \frac{n+1}{n+2}$$

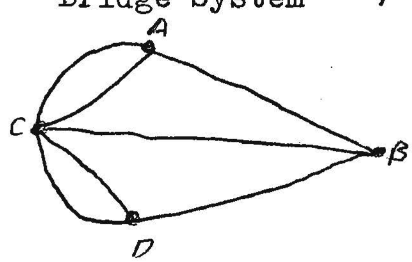
Proved for any n

The next chapter delt with what topology was. It began describing Euclidean Geometry and then told how topology was different. Most descriptions of geometry you can use also in topology. The main difference is the movements in geometry are rigid motions, in which the distance between any two points is not changed. In topology elastic motions are allowed. We can imagine that our figures are made out of perfectly elastic rubber. We can cut these figures but must put points that were together back together. Two figures are topologically equivalent iff one figure can be made to coincide with the other by an elastic motion. A circle and a square would be topologically equivalent. Also, a sphere with one handle and a doughnut would be topologically equivalent.

Chapter three explained networks and maps and introduced these subjects with the Königsberg question and the colors needed for a map. The Königsberg question is this: Could you take a walking tour of Köningsberg and only cross each of the seven bridges just once?



Bridge System



System reduced to seven arcs
and four points

To answer this question we must determine a network, a figure consisting of a finite non-zero, number of arcs (line segments) no two of which intersect except possibly at their end points, called verticies. These verticies are either even or odd depending on the number of arcs comming into them. Since in the Koingsberg question the network can be drawn with four odd verticies then using the theorem that; If a network has more than two odd verticies, it cannot be traversed by a single path. There were a lot of theorems derived about paths and networks using facts about verticies and their properties pertaining to arcs. I won't go into these. A network is connected iff every two different vertices of the network are vertices of some path in the network. A map is a network, together with a surface which contains the network. An important theorem proved here was the Euler Theorem: If V, E, and F are, respectively, the numbers of vertices, edges, and faces of a connected planar map, then $V - E + F = 2$.

Proof: It is intuitively evident that any connected map in a plane can be built up by starting with a single edge and performing a succission of the following three operations.

(i) Add a new edge joined at one end only;
added: 1 vertex, 1 edge, no faces.

(ii) Add a new vertex in an existing edge;
added: 1 vertex, 1 edge, no faces.

(iii) Add a new edge joined at both ends;
added: no vertices, 1 edge, 1 face.

When we start with just one edge there are two possibilities; either there are two vertices and one face or there are only one vertex and two faces; in either case,

$$V - E + F = 2.$$

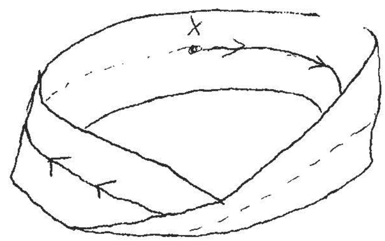
Now notice that none of the three operations described above makes any change in the sum $V - E + F$ since each adds one edge and either adds a vertix and no faces or adds a face and no vertices. Thus, with V, E, and F being, respectively, the number of vertices, edges, and faces in the completed map, we still will have

$$V - E + F = 2.$$

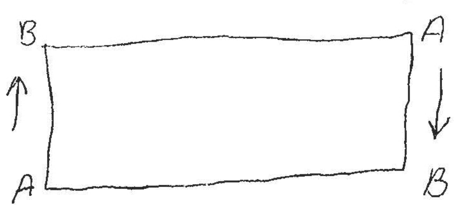
Using this, it can be proved that any map in a plane can be colored with five colors. No one has found an example of a planer map which requires five colors. In each example that has been examined, it has been possible to color the map with only four colors. This four color proof is still the question for much mathematical research.

Chapter four expanded the idea of topological equivalence to three-dimensional space. "Openedness" and "closedness" was discussed, and an example will explain the idea: A sphere is a surface in three dimensions, all points of which are at a given distance from some particular point, the radius. An open ball is the portion of three-dimensional space which is enclosed within some sphere, but not including the sphere. A closed ball is the position inside or on the same sphere. This can also be used to describe opened and closed disks, circles, toruses, ect. This relates to opened and closed sets in point-set topology.

In classifying these surfaces we use certain new terms. A manifold is a connected surface (all in one piece) such that, sufficiently near to each point, the surface is topologically equivalent to an open disk. A surface is bounded iff the entire surface is contained in some open ball. A boundary of a particular piece of a surface is the curve which separates that piece from the rest of the surface. A lot of these surfaces and manifolds can be represented as rectangles. Two examples of one sided surfaces were given. The Möbius strip and the Klein bottle, the latter not being able to be formed in three-dimentional space.



Möbius Strip



Rectangular Representation
(Möbius Strip)

If a fly was to start at point x and walk around the Möbius strip he would end up on the other side without crossing any edges so he is really still on the same side and it is a one sided figure.

Maps were also expanded to cover these three-dimensional surfaces. Euler's theorem was proved for a sphere by using a polar projection of it into a connected map in a plane, which with the other proof proves that for a map on a sphere

$$V - E + F = 2$$

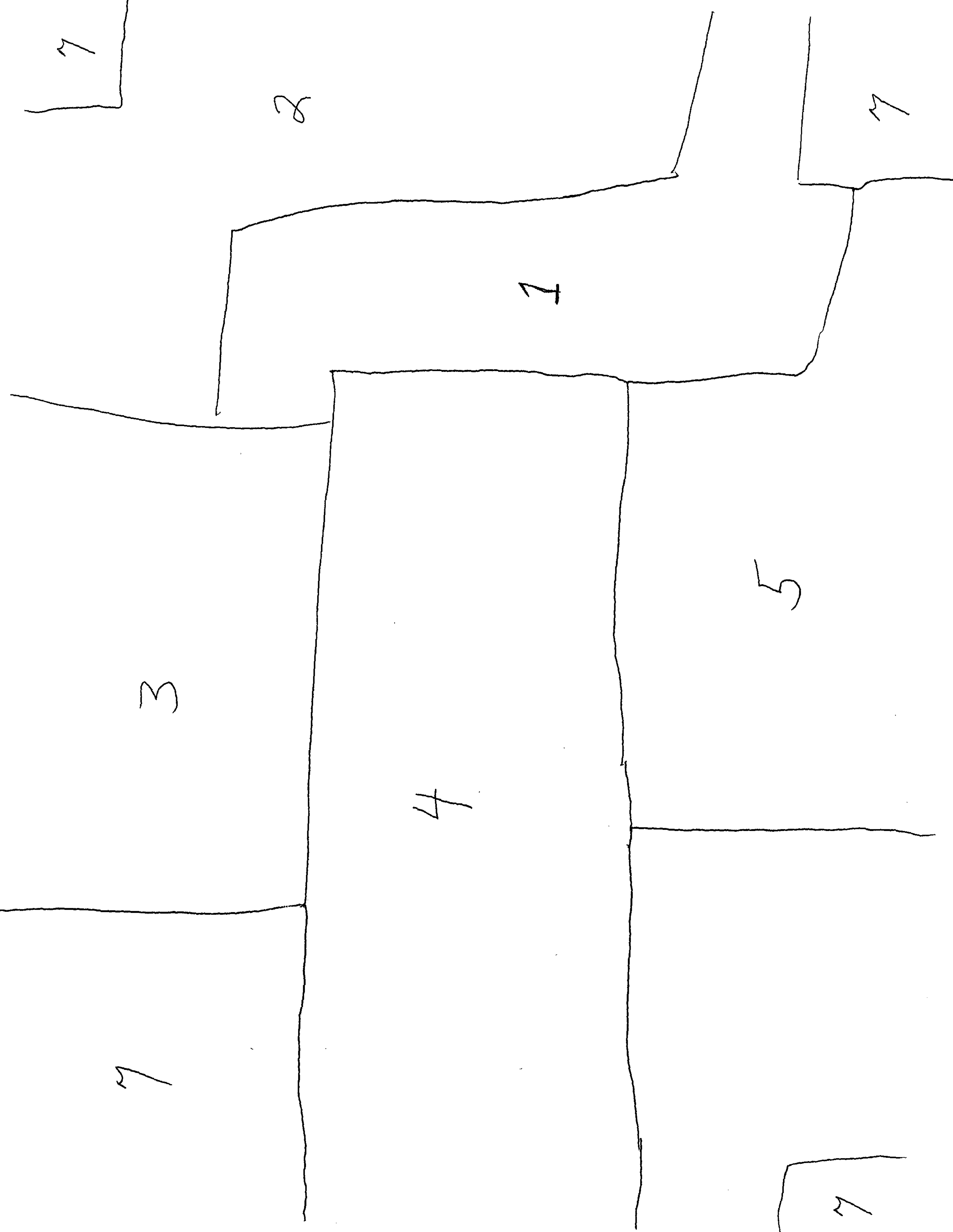
If the sphere has p handles with V vertices, E edges, and F faces, and if each is simply connected, then

$$V - E + F = 2 - 2p$$

This is proved much like Euler's theorem. A torus requires seven colors and this can be proven. The last page is an example of a torus drawn like a rectangle that would require seven colors.

The last thing we covered was the Jordan Theorem which was beyond the scope of the book for the exact proof. It states: If S is any simple polygon in a plane P, the points of P which are not on S can be divided into two sets A and B in such a way that any two points in the same set can be joined by a polygonal path not intersecting S, while no two points, one of which is in A and the other in B, can be so joined.

My conclusions about topology are that it would have very little value toward physics as say calculus and geometry and the other forms of math. Point set topology is very useful in the study and understanding of sets but has little physical meaning. This study has been fun and interesting and has raised some questions that couldn't be answered.


3
4
5