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The Development of the Calculus

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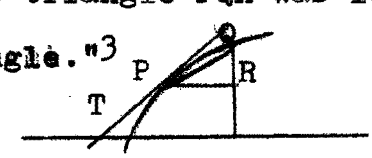
The Greeks made the first step in the inquiry of the infinitely small quantities by an attempt to determine the area of curves. The method of exhaustions they used for this purpose consisted of making the curve a limiting area, to which the circumscribed and inscribed polygons continually approached by increasing the number of their sides. The area obtained was considered to be the area of the curve. The method of integration is somewhat similar, to the extent that it involves finding the limits of sums. Zeno of Elea (c. 450 B.C.) was one of the first to work with problems that led to the consideration of infinitesimal magnitudes, and Leucippus (c.440 B.C.) and Democrites (c. 400B.C.) taught that magnitudes are composed of indivisible elements in infinite numbers. Archimedes' (c. 225 B.C.) work was the nearest approach to actual integration among the Greeks: his first noteworthy advance was to prove that the area of a parabolic segment is 4/3 of the triangle with the same base and vertex, or 2/3 of the circumscribed quadrilateral. He also anticipated many modern formulas in his treatment of solids bounded by curved surfaces.[1]

There are only traces of an approach to the calculus in the Middle Ages, and Pappus of Alexandria (c. 390), who followed Archimedes' work, contributed the most from the time of Archimedes until the seventeenth century. During the first half of the seventeenth century, methods of limited scope began to appear for constructing tangents, determining maxima and minima, and finding areas and volumes. Few general rules were developed, but the essential ideas of the derivative and definite integral were beginning to be formulated. Kepler's study of planetary motion demanded some method for finding areas of sectors which he called "sum of the radii", a crude kind of integration; he also considered solids as composed of infinitely many infinitely small cones of thin disks, whose summation became the problem of later integration. Roberval considered the area between a curve and a straight line as made up of an infinite number of infinitely narrow rectangular strips, the sum of which gave him the required area. Fernat's work was similar, and his methods for obtaining maxima and minima and for drawing tangents to curves had such striking resemblances to those of the differential calculus that Laplace and Lagrange pronounced him to be the inventor.[2] Barrow in his _Lectiones opticae et geometricae_ gave a method of tangents in which Q approaches P as in our present theory, the result being and infinitely small arc. The triangle PQR was long known as "Barrow's Differential Triangle."[3]



"There are certain focal points in history toward which the lines of past progress converge, and from which radiate the advances of the future. Such was the age of Newton...."[4]

The early seventeenth century mathematicians bent the force of their genius in a direction which eventually led to the discovery of the infinitesimal calculus by Newton and Leibniz with the help of the new geometry.

The ancients had considered the area of a rectangle as produced by the motion of one of its sides along the other; Newton extended this principle to all kinds of mathematical quantities. All kinds of figures can be described by the motion of bodies, but quantities generated in this manner in a given time become greater or less, in proportion as the velocity with which they are generated is greater or less. This is the consideration that led Newton to apply himself to finding out the magnitudes of finite quantities by the velocities of their generating motions and that gave rise to the method of fluxions before he was twenty-four years old.[5] "Having met with an example of the method of Fermat, M.Newton succeeded in applying it to adfected equations, and determining the proportion of the increments of indeterminate quantities."[6] These increments he called moments; the velocites with which the quantities increase he called motions, velocities of increase, and fluxions; and he applied the name flowing quantities to all quantities which increase in time.

Newton's analysis, consisting of the method of series and fluxions combined, was so universal as to apply to almost all kinds of problems. He not only invented the method of fluxions in 1665, in which the motions or velocities of flowing quantities increase or decrease, but he also considered the increase or decrease of these motions themselves, to which he later gave the name of second fluxions He extended his newly discovered method to include the functions then in common use, recognized the fact that the inverse problem of differention could be used in solving the problem of quadrature, and developed a wide range of applications.[7]

The quantities considered by Newton as gradually and indefinitely increasing, fluents or flowing quantities, he represented by the letters v, x, y, and z; quantities known and determinate he represented by a, b, c, d; and the velocities by which every fluent is increased by its generating motion he represented by $\dot{v}$, $\dot{x}$, $\dot{y}$, $\dot{z}$. In <u>The Method of Fluxions</u>, translated by J. Colson from Newton's Latin, Newton considered two problems concerning a space described by local motion, however accelerated or retarded: 1) "The length of the space described being continually given; to find the velocity of this motion at any time proposed, 2) The velocity of the motion being continually given; to find the length of the space described at any time proposed."[8] The first problem is equivalent to differentiation; and the second to integration, which Newton termed the method of quadrature, or to the solution of a differential equation, which Newton called the inverse method of tangents.[9]

Newton solved the first problem by the following method:

> "Dispose the equation, by which the given relation is expressed, according to the dimensions of some one of its flowing quantites, suppose x, and multiply its terms by any arithmetical progression, and then by $\dot{x}/x$; and perform this operation separately for every one of

> the flowing quantities. Then make the sum of all the products equal to nothing and you will have the equation required."[10]

If the relation of the flowing quantities is $x^3 - ax^2 + axy - y^3 = 0$, first dispose the terms according to the powers of x, then y, and then multiply them in the following manner:

$x^3 - ax^2 + axy - y^3$	$-y^3 + axy - ax^2 + x^3$
Multiply by:	Multiply by:
$\frac{3\dot{x}}{x} \cdot \frac{2\dot{x}}{x} \cdot \frac{\dot{x}}{x} \cdot 0$	$\frac{3\dot{y}}{y} \cdot \frac{\dot{y}}{y} \cdot 0$
$3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y$	$-3\dot{y}y^2 + a\dot{y}x$

The sum of the two products is $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0$, which gives the relation between the fluxions $\dot{x}$ and $\dot{y}$. If the proposed equation contained complex fractions or "surd" quantities such as $\sqrt{a^2 - x^2}$, Newton substituted a letter for them and proceeded as in the above example.

In respect to the second problem, which is equivalent to integration, Newton divided equations into three different cases: 1) Those in which two fluxions of quantities and only one of their flowing quantities are involved, 2) those in which two flowing quantities are involved together with their fluxions, and 3) those in which the fluxions of more than two quantities are involved. So that the flowing quantities might be more easily distinguished from one another, the fluxion that is put in the numerator of the fraction which indicates the ratio of the fluxions is called the "relate quantity" and the one in the denominator the "correlate."

Solution of Case I: The flowing quantity which is contained in the equation is assumed to be the correlate, and the ratio of the fluxions is equal to a quantity in terms of this correlate. First multiply the value of the ratio of the fluxions by the correlate quantity, then divide each of its terms by the power to which it is raised. What results will be equivalent to the other flowing quantity. Let the equation be $\frac{\dot{y}}{\dot{x}} = a - \frac{x}{4} + \frac{x^2}{64a} + \frac{131x^3}{512a^2}$; then the correlate quantity is in terms of x. Multiply $a - \frac{x}{4} + \frac{x^2}{64a} + \frac{131x^3}{512a^2}$ by x, and the result is $ax - \frac{x^2}{4} + \frac{x^3}{64a} + \frac{131x^4}{512a^2}$. After dividing each term by the power of the corresponding x term and equating to y, the result is $y = ax - \frac{x^2}{8} + \frac{x^3}{192a} + \frac{131x^4}{2048a^2}$.

Solution of Case II: For this solution, the equation must be changed to one involving the ratio of the fluxions equated to any aggregate of simple terms without any fractions denominated by the flowing quantity. Let the equation be $\dot{y}/\dot{x} = 1 - 3x + y + x^2 + xy$. The terms $1 - 3x + x^2$ (which are not affected by the relate quantity y) are written in the table as shown, and the rest of the terms, y and xy, are written in the left column. After doing this, multiply the first tern of the correlate quantity by the correlate, x, giving x, and then divide by the number of dimensions, 1, giving x. Substituting x for y in the marginal terms y and xy gives x and x^2 which are written to the right of these terms in the table. The next least terms $-3x$ and x are added, and the process is continued <u>in infinitum</u>. When the sum is obtained, then it is acted upon as though it were an equation of Case I and y is obtained.

	$1 - 3x + x^2$
y	$x - x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5$, etc.
xy	$x^2 - x^3 + \frac{1}{3}x^4 - \frac{1}{6}x^5 + \frac{1}{30}x^6$
sum	$1 - 2x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{4}{30}x^5$, etc.
$y = x - x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5 - \frac{1}{45}x^6$, etc.	

Solution of Case III: If an equation involves three of more fluxions of quantities, any relation between any two of these quantities may be assumed, and the relation of the fluxions can be found accordingly. Let the proposed equation be $2\dot{x} - \dot{z} + \dot{y}x = 0$. Assume $x = y^2$, therefore $\dot{x} = 2y\dot{y}$, and substitute into the original equation: $4y\dot{y} - \dot{z} + \dot{y}y^2 = 0$. Using Case I to solve this equation yields $2y^2 + \frac{1}{3}y^3 = z$, and by substituting x for y^2 and $x^{\frac{3}{2}}$ for y^3, $2x + \frac{1}{3}x^{\frac{3}{2}} = z$ results.

Besides the solution of these two problems in the <u>Method of Fluxions</u>, Newton determines maxima and minima, the radius of curvature of curves, and other geometrical applications of his fluxionary calculus. The method employed is strictly infinitesimal. The fundamental principles of the fluxionary calculus were first given to the world in Newton's <u>Philosophiae Naturalis Principia</u> (1687), but its notation did not appear until 1693 in the second volume of Wallis' <u>Algebra</u>. The exposition given in the <u>Algebra</u> was contributed by Newton and rests on infinitesimals, as does the first edition of Principia. However, in the second edition the foundation is somewhat altered, and in the Quadrature of Curves (1704) the infinitely small quantity is completely abandoned. Thus, it appears that Newton's doctrine was different in different periods.[12]

"The method of limits is frequently attributed to Newton, but the pure method of limits was never adopted by him as his method of constructing the calculus."[13] He established in his Principia certain principles which are applicable to that method, but used them for a different purpose. The first lemma of the first book has been made the foundation of the method of limits:

> "Quantities and the ratios of quantites, which in any finite time converge continually to equality, and before the end of that time approach nearer the one to the other than by any given difference, become ultimately equal."[14]

Gottfried Wilhelm Leibniz, the second and independent inventor of the calculus, during visits to France and England in the 1670's on political or diplomatic missions, met the leading French and English men of science, and in exchange for some of their ideas disclosed his own. In this way he learned about contemporary advances in algebra and geometry, especially from Henry Oldenburg and Huygens. Soon he discovered the fundamental principle of the calculus: that differentiation, the means of studying limits and rates, is the inverse of integration. In the hands of Leibniz, the differential calculus made rapid progress. In the Acta Eruditorum, which appeared at Leipsic in October, 1684, he published the more important parts of his study of the quadrature of curves. In 1686 a paper containing the rudiments of the integral calculus was published in which he treated the quantities dx and dy as infinitely small and showed that by the use of his notation properties of corves could be fully expressed.[15] The early distinction between the systems of Newton and Leibniz lies in the fact that Newton used the infinitely small increment as means of determining velocity of fluxions, while Leibniz considered the relation of the infinitely small increments as itself the object of determination. The difference rests upon a difference in the manner of generating quantities.[16]

"Unlike most mathematicians of his day,....(Leibniz) made an extended study of notation.... The notation of the calculus as we know it is in large part due to Leibniz."[17] He proposed to represent the process of intgration by the symbol $\int$, the old form of s, signifying "summation" and to represent the inverse operation by d. By 1675, he had settled this notation, writing $\int ydy = \frac{1}{2}y^2$ as it is written today. He spoke of the integral calculus as the <u>calculus summatorius</u>, and in 1669 he adopted the term <u>calculus integralis</u>, already suggested by Jacques Bernoulli in 1690.? Newton used dots and dashes above the letters to indicate "fluxions" and "fluents", but they were difficult to read and to print.

> "It is generally agreed that the development of the calculus in England was hindered until well into the nineteenth century because English mathematicians remained loyal to Newton's notation while their continental colleagues moved ahead into new areas with Leibniz' more expressive system."[18]

Leibniz and Newton became embroiled in a bitter struggle over which of them had first devised the calculus. Newton firmly believed that Leibniz had derived the differential calculus from papers actually communicated to him or from his ideas which were in circulation at the time of Leibniz's visit to London in 1673. Dispute between the friends of both Newton and Leibniz led to a report by a special committee of the Royal Society which influenced English readers of the eighteenth century to give Leibniz little credit. It is now fairly certain that each discovered the calculus independently. Newton wrote on his method of fluxions as early as 1665 but did not publish on the subject until 1687, three years after Leibniz had published in the journal Acta Eruditorium a brief essay which proceeded on different lines from Newton's work and used original symbolism.[19] Neither Leibniz nor Newton, however, was able to establish a rigorous basis for the calculus, but both overcame the obstacle set up by the ancient mathematicians: the belief that scientific treatment of variability was impossible because of the unchanging nature of true reality.[20]

The general trend from 1700 to 1900 was toward a stricter arithmetization of three basic concepts of the calculus: number, function, limit. In the first and crudest stage of this period, Thomas Simpson (1737-1776, Eng.) attempted to clarify, in his Treatise on fluxions, Newton's intuitive approach to fluxions through the generation of "magnitudes" by "continued motion," but only succeeded in adding deeper obscurity. Continental mathematicians at the same time, followed the tradition of Leibniz as handed down by John Bernoulli in 1691-2 to l'Hospital. They proceeded from the mystical doctrine that "a quantity which is increased or decreased by an infinitely small quantity is neither increased nor decreased," and this became the age of the"little zero."

During the period from 1730 to 1820, L. Euler, J. Lagrange and P. S. Laplace developed higher analysis and severed it completely from geometry. Euler brought about an emancipation of the analytical calculus from geometry and established it as an independent science. He developed the calculus of finite differences in the first of his Institutuiones calculi differentialis (1775) and then deduced the differential calculus from it. His research on series led to the creation of the theory of definite integrals by the development of the so-called "Eulerian integrals." There are few great ideas pursued by succeeding analysts which were not suggested by Euler. At the age of nineteen J. Lagrange communicated to Euler a general method of dealing with "isoperimetrical problems", known as the calculus of variations. Lagrange did quite as much a Euler towards the creation of the calculus of variations, but instead of assuming the limits of the integral as fixed, he allowed all co-ordinates of the curve to vary at the same time. In 1766 the name "calculus of variations" was introduced by Euler, and he did much to improve this science along the lines marked out by Lagrange. Laplace applied the calculus of Newton and Leibniz, with mechanics, to the elaboration of Newton's theory of gravity.[22]

> "It is generally agreed that reasonably sound but not necessarily final ideas of limits, continuity, differentiation and integration came only in the nineteenth and twentieth centuries, beginning with Cauchy in 1821-3."[23]

The definition of limit and continuity current today in texts on elementary calculus are basically those of Cauchy used in his lectures and writings. He defined the differential quotient, or derivative, as the limit of a difference quotient, the definite integral as the limit of the sum, and differentials as arbitrary real number.[5] The continuity of a function and the convergence and divergence of an infinite series are referred to the concept of a limit. G. F. B. Riemann (1826-1866, German) in 1854 investigated the representation of a function by a trigonometric (Fourier) series. He discovered that Cauchy had been too restrictive in his definition of an integral: he showed that definite integrals of sums exist even when the integrand in discontinuous. Like Cauchy and Riemann, other mathematicians since the time of Newton and Leibniz have improved and added to the calculus; in this way the calculus continues in its development.

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[2]Ibid.

[3]Smith, op. cit., pp. 690-691.

[4]Florian Cajori, A History of Mathematics (New York: Macmillan Co., 1961), p. 191.

[5]Derek T. Whiteside, The Mathematical Works of Isaac (New York: Johnson Reprint Corporation, 1964), p. 33; and E. T. Bell, The Development of Mathematics (New York: McGraw Hill Book Co., Inc., 1945), p. 146.

[6]Brewster, op. cit., p. 11.

[7]Ibid., p. 14; and Smith, op. cit., pp. 692-693.

[8]Whiteside, op. cit., p. 48.

[9]Smith, op. cit., pp. 695-696.

[10]Whiteside, op. cit., pp. 49-50.

[11]Ibid.,p. 50-66.

[12]Cajori, op. cit., p. 196.

[13]Ibid., p. 199.

[14]Ibid..

[15]Fredierick C. Kreiling, "Leibniz," Scientific American, 218:98, May 1968; Brewster, op. cit, pp. 27-28; and Bell,op. cit., p. 149.

[16]Cajori, op. cit.,p. 196.

[17]Kreiling, op. cit., p. 95-96.

[18]Ibid.

[19]Ibid., p. 98; and Brewster, op. cit., p. 75.

[20]Kreiling, op. cit., p. 98.

[21]Bell, op. cit., pp. 283-284.

[22]Cajori, *op. cit.*, pp. 231-239, 251.

[23]Bell, *op. cit.*, p. 153.

[24]*Ibid.*, p. 292-293.

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