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Mathematics and Logic

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Mathematics is interested in the methods by which concepts are defined in terms of others and statements are inferred from others. It therefore uses a primarily deductive form of reasoning. It is almost impossible to distinguish where logic leaves off and mathematics begins. "... logic is the youth of mathematics and mathematics is the manhood of logic."[1] Mathematics starts from certain premises and, by a strict process of deduction, arrives at the various theorems which constitute it.

> All traditional pure mathematics, including analytical geometry, may be regarded as consisting wholly of propositions about the natural numbers. That is to say, the terms which occur can be defined by means of the natural numbers, and the propositions can be deduced from the properties of the natural numbers - with the addition in each case, of the ideas and propositions of pure logic.[2]

In order to understand the congruence of mathematics and deductive logic, one must understand the principles of each and the relation between them. The ancients called logic the instrument of science and considered it as a preparatory to all science.

Logic is the science and art of right thinking. It does not deal with reality but only with the operation of thinking itself. Reason is a form of mental activity which enables one to deal with new situations with novel data. An inference from a law or general principle to some consequence from its relation to another principle is known as a deductive inference. Deductive reasoning argues from the universal to the particular,

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[1]Bertrand Russel, Introduction to Mathematical Philosophy (London: George Allen and Unwin Ltd., 1956), p. 194

[2]Ibid., p. 4.

or from the more to the less universal, by way of a middle term. A term is a statement which expresses a concept of simple apprehension. When a term indicates the entire class for which it stands it is said to be distributed. When it represents only a part of the class it is undistributed. The act by which the mind affirms or denies a term is a judgment. In deduction, two judgments which have a common term between them are related so that a third judgment is necessarily implied from them.

Deductive reasoning has two aspects. First, it is an act of the mind. Second, it is a definitely structured combination of verbal symbols or words. The verbal statement is called the syllogism. The act of the mind is the deductive inference. Therefore, a syllogism is the expression of the mental act of deduction. Propositions are expressions of judgments made by the mind. These propositions are usually distinguished according to whether they are affirmative or negative and universal or particular. The syllogism contains three propositions. The two propositions which imply the third are called the antecedents or premises. The implied proposition is the consequent or conclusion. All syllogisms must follow certain rules in order to be logically correct. (1) Every syllogism contains three propositions. (2) Each syllogism contains three and only three terms. (3) The middle term must be distributed at least once. (4) The middle term must not occur in the conclusion. (5) No term may have a greater distribution in the conclusion than it had in the premise. (6) Two negative premises will not yield a conclusion. (7) If one premise is negative, the conclusion must be negative; if both premises are affirmative, the conclusion must be affirmative. (8) If one premise is particular, the conclusion is particular; if both premises are particular, there is no conclusion.

What may at first appear to be a logical conclusion may upon examination be false. If a syllogism does not comply with all the rules of a syllogism, it may render a false conclusion.

For example: All dogs eat meat.
Joe eats meat.
∴Joe is a dog.

Joe may or may not be a dog. The conclusion is logically incorrect according to rule (3). The middle term, meat, is not distributed in either premise.

Behind all deductive reasoning lies the principle called "Dictum de Ommi et Nullo" which states: "Whatever statement may be made with regard to a class taken generally may be made of each and every member of that class."[3] The major premise of the syllogism asserts that the whole of a certain class is included in another class or is excluded from it. The minor premise asserts that certain things are included in the first class. The conclusion applies the things asserted in the minor premise to the assertion made in the major premise.

For example: All cats have claws.
The tiger is a cat.
∴The tiger has claws.

"All cats have claws" is the major premise and asserts that all cats are included in the group of things which have claws. The minor premise, "The tiger is a cat," asserts that "the tiger" is included in the class of cats. The conclusion applies the assertion of the minor premise, that the tiger is a cat, to the assertion of the major premise, that all cats have claws, and concludes that tigers have claws.

Propositions of a syllogism are classified according to quality and quantity. Propositions which are universal and affirmative are termed

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[3]Adam Leroy Jones, Logic, Inductive and Deductive (New York: Henry Holt and Company, 1909), p. 127.

"A" propositions. Universal propositions which are negative are "E" propositions. Those which are particular and affirmative are "I" propositions. Particular and negative propositions are "O" propositions. Propositions which differ in quantity but not quality are known as subalterns. The A proposition is the subaltern of the I proposition and E is the subaltern of O. One is particular while the other is universal. If the universal is true, then the particular is true. However, if the particular is true the universal may or may not be true. Contraries are opposite propositions. They are universals which differ in quality but not in quantity. Subcontraries are particulars which differ in quality but not quantity. A proposition and its contrary cannot both be true for they deny each other; they both, however, may be false. A and E propositions are contrary to each other. Subcontraries may both be true, both be false, or one true and the other false. I and O propositions are subcontraries of each other. Contradictories are propositions which differ in both quality and quantity. Contradictories cannot both be true or both be false. One must be true and the other must be false. A and O propositions and E and I propositions are contradictories. The relationships of subalterns, contraries, subcontraries and contradictories are diagramed in the "Square of Opposition."

Every man is happy. --- Contraries ------ No man is happy.
All x is y. No x is y.

Subalterns Contradictories Contradictories Subalterns

Some man is happy. ---- Subcontraries - Some man is not happy.
Some x is y. Some x is not y.

There are different ways of stating the same fundamental truth. The negative statement of an original proposition is known as the obverse.

Proposition: Every a is b.
Obverse: No a is not b.

The converse of a proposition is the interchanging of subject and predicate while taking into consideration the quantity of each term.

Proposition: Every a is b.
Converse: Some b is a.

O propositions have no converse. The converse of the obverse yields the contrapositive.

Proposition: Every a is b.
Obverse: No a is not b.
Contrapositive: Some non b is not a.

A proposition must tell something about the nature of a subject or it cannot be used as a basis for inferring anything about other subjects which have that nature. All x's by examination may be y's, but unless y relates in some way to x the next x may or may not be a y. This type of reasoning which has no basic relationships between the terms can never give rise to a genuine deductive inference. A conclusion based on observation and experimentation is known as inductive logic. An inductive argument is built on a set of statements that are taken to be facts or truths. The facts of one inductive argument may be the conclusions of earlier arguments. One argument builds upon another. Deductive reasoning argues from the universal to the particular; whereas, inductive reasoning goes from the particular to the universal. The facts of inductive logic arise from observation, experimentation, and previous conclusions. Deductive reasoning must be based on a previous inductive conclusion; for nothing can be deduced from nothing. The truth of a deductive argument is based on the truth of the premises which it contains.

For example: All tress have green leaves.
The oak is a tree.
The oak has green leaves.

Before the conclusion that oaks have green leaves can be determined and have some meaning, the facts, all trees have green leaves and the oak is a tree, must be established. These are established by observation or inductive reasoning.

All language consists of signs or symbols. "A sign is an arbitrary mark, having a fixed interpretation, and susceptible of combination with other signs in subjection to fixed laws dependent upon their mutual interpretation."[4] All the operations of language as an instrument of reasoning may be indicated by the use of a system of signs and symbols. Literal symbols such as x, y, and z may be used to represent the subjects of the conceptions of the mind and the characteristics belonging to such subjects. Symbols such as +, −, and = represent the operations of the mind by which the subjects and predicates are combined. These symbols of logic are subject to definite laws which partly agree and partly disagree with the laws of the corresponding symbols in algebra. The laws of the symbols and those of the mental process of logic are identical.

The mind may think of a subject. This is represented by an appropriate symbol such as x, y, or z. The mind, however, may not think of a single subject but a group of subjects consisting of partial groups, each of which is separately named or described. In thinking the conception of a group consisting of partial groups, the subjects are connected by "and" or "or". In algebra and logic, the words "and" and "or" are analogous with the sign "+". The statement "Trees and minerals" would be represented

[4]George Boole, An Investigation of the Laws of Thought (New York: Dover Publications, Inc., 1854), p. 25.

symbolically as $x + y$ with x representing trees and y representing minerals. Since it is possible to collect into a whole it is also possible to separate a part from a whole. The separation of a part from a whole is usually expressed by the term "except". This operation is experessed symbolically by "-".

Example of negation:

Let x = animals
y = feathers
xy = animals with feathers
$\therefore x - xy$ = All animals except animals with feathers

Anything which is characteristic of each member of a group formed by partial groups is the same as if the characteristic were first possessed by each member of the partial groups. Therefore, the result of the previous example, $x - xy$, may be written $x(1 - y)$ which is read "Animals, all except those with feathers."

The copula is that which connects subject and predicate. It either implies or is some form of the verb "to be" and is expressed symbolically by "=".

Let x = stars
y = suns
z = planets
$\therefore x = y + z$ The stars are the suns and the planets.
or $x - z = y$ The stars except the planets are suns.

If two classes of things, x and y, are identical, then the members of one class which possess a given property will be identical with those members of the other class which possess the same property.

If $x = y$
Then $zx = zy$

This is analagous to the algebraic law which states, if both members of an equation are multiplied by the same quantity, the products are equal. The order of terms in an equation does not matter. In the conception of "good men" it matters not whether the conception of the group of men is conceived and then limited to those which are good, or whether the first conception is that of the group of things which are good and then limited to those good things which are men. Therefore, $xy = yx$. This is equivalent to the commutative property in algebra.

The combination of two literal symbols in the form xy expresses the entire class of objects which possess the qualities represented by x and y. If the two symbols have exactly the same meaning, their combination expresses no more than either of the symbols alone expresses.

If $xy = x$ and $x = y$
then $xx = x$
or $x^2 = x$

The equation $x^2 = x$ can have no other roots than 0 and 1. Since $x^2 = x$ is a characteristic of logical symbols, logical symbols would be equivalent to the symbols of quantity using only the values 0 and 1. 0 symbolizes the class representing "nothing". No matter what y may represent, the things which belong to it and to the class "nothing" are identical with those included in the class "nothing" or $0y = 0$. The symbol 1 satisfies the law $1y = y$. The symbol 1 must represent a class which is equivalent to all the members common to any proposed class y and itself. Therefore the class 1 must be the "universal" class since it is the only class in which are found all things which exist in any class. Representing any class of objects by x, $(1 - x)$ will represent the contrary class of objects. The universal class, excluding those things in the class of x, is everything that is non-x. The equation $x^2 = x$ may be changed to $x(1 - x) = 0$ which is known as the law of duality. The equation states that a class which contains all members of the class x and no members of the class x does not exist.

The following example demonstrates the principles previously discussed.

x = hard, y = elastic, z = metals
Hard, elastic, metals = xyz
Non-elastic hard metals = $xz(1 - y)$
Elastic substances and hard non-elastic metals = $y + xz(1 - y)$
Hard substances, except metals = $x - z$
Metalic substances except those which are neither hard nor elastic = $z - z(1 - x)(1 - y)$

When either the subject or the predicate of a primary proposition is particular, the indefinite class simbol, "v", is used to designate such. In considering the proposition,"All men are mortal," it is clear that the meaning is all men are some mortal beings. Let y represent "men" and x represent "mortal beings." The expression would then be $y = vx$ with v showing that x is particular rather than universal. To form the symbolical expression of any primary proposition, form the expression of the subject and that of the predicate and then equate the resulting expressions. To express a negative proposition, convert it into the form all x's are non-y's and then proceed to equate the expression.

No men are perfect beings.
All men are non-perfect beings.
y = men, x = perfect beings
$y = v(1 - x)$

Valid reasoning by the use of symbols must follow certain conditions. (1) A fixed interpretation must be assigned to the symbols employed in the expression, and the laws of the combination of these symbols must be correctly determined from the interpretation. (2) The formal process of solution must be conducted in obedience to the laws determined without regard to the question of the interpretability of the results obtained. (3) The final result must be in interpretable form, and be interpreted in accordance with that system of intertation which has been employed in

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the expression of the proposition. When any function, $f(x)$, in which x is a logical symbol, or a symbol of quantity susceptible only to the values of 0 and 1, is reduced to the form $ax + b(1 - x)$, a and b being determined as to make the result equivalent to the function, the function is said to be developed.

To develop the function $f(x)$

$$f(x) = ax + b(1 - x)$$
$$\text{if } x = 1 \quad f(1) = a$$
$$\text{if } x = 0 \quad f(0) = b$$
$$\therefore f(x) = f(1)x + f(0)(1 - x)$$

To develop a function involving any number of logical symbols, the function is developed as a function of each of the symbols alone keeping their relation to each other.

Develop $f(x,y)$

$$f(x,y) = f(1,1)xy + f(1,0)x(1 - y) + f(0,1)(1 - x)y + f(0,0)(1 - x)(1 - y)$$

$$\text{if } f(x,y) = \frac{1 - x}{1 - y}$$

$$f(1,1) = \frac{0}{0} \quad f(1,0) = 0 \quad f(0,1) = \frac{1}{0} \quad f(0,0) = 1$$

$$f(x,y) = \frac{0}{0}xy + 0x(1 - y) + \frac{1}{0}(1 - x)y + (1 - x)(1 - y)$$

The development of any expression, $f(x)$, consists of two terms, x and $1 - x$, mulitplied by the coefficients $f(1)$ and $f(0)$ respectively. The terms are referred to as the constituents, and the coefficients as the factors. To develop any function of any number of terms, $f(x,y,z,\ldots)$, form a series of constituents by letting the first constituent be the product of the symbols. Change in this product any symbol, z, into $(1 - z)$ for the second constituent. Then in both these change another symbol, y, into $(1 - y)$ for two more constituents. Then in the four constituents obtained change the next symbol, x, into $(1 - x)$ for four more constituents. Proceed in this manner until the number of possible changes is exhausted.

To find the coefficient of any constituent, change the term x in the original function to 1 if the constituent involves x, or into 0 if the constituent contains $(1 - x)$ as a factor. Apply the same rule to the other symbols and the coefficients are obtained.

After an expression has been developed, the constituents represent all the classes of objects which can be descrived by the affirmation or denial of the properties expressed by x and y. There is no object which can not be described by the presence or absence of some property; thus each thing in the universe may be referred to by the possible combinations of the given classes and their contraries. The symbol 1 as the coefficient of a term indicates that the class which that constituent represents exists according to the original equation. Classes which are not true or do not exist under the conditions of the expression have a coefficient of 0. The symbol $\frac{0}{0}$ as a coefficient indicates that an indefinite portion of the class exists. The indefinite class symbol v may be subsituted for $\frac{0}{0}$. Any other symbol as a coefficient indicates that its constituent must be equated to 0 to obtain its meaning.

Taking the expression:

Responsible beings are all rational beings who are either free to act, or have voluntarily sacrificed their freedom.

Let x = responsible beings
y = rational beings
z = those who are free to act
w = those who have voluntarily sacrificed their freedom

Equating the expression:

$$x = yz + yw$$

Determine the relationship of rational beings:

$$y = \frac{x}{z + w}$$

Developing and rejecting terms whose coefficients are 0:

$$y = \tfrac{1}{2}xzw + xz(1 - w) + xw(1 - z) + \frac{1}{0}x(1 - z)(1 - w) + \frac{0}{0}(1 - x)(1 - z)(1 - w)$$

Equating to 0 terms whose coefficients are $\frac{1}{2}$ and $\frac{1}{0}$:

$$y = xz(1 - w) + xw(1 - z) + v(1 - x)(1 - z)(1 - w)$$

$$xzw = 0$$

Conclusion:

Rational beings are all responsible beings who are either free to act, or have voluntarily sacrificed their freedom, and an undetermined number of beings not responsible, not free, and not having voluntarily sacrificed their freedom. No responsible beings are free to act and have sacrificed their freedom.

Every syllogism states the identity of two terms because of their identity with a third term. This process is analagous to and the basis of the mathematical substitution of equals for equals.

Taking the previous example and saying:

All responsible beings are dependable.
Let t = dependable beings

Subsituting:

$$y = tz(1 - w) + tw(1 - z) + v(1 - t)(1 - z)(1 - w)$$

Conclusion:

Rational beings are dependable beings who are either free to act, or have voluntarily sacrificed their freedom, and an undetermined number of beings not dependable, not free, and not having voluntarily sacrificed their freedom.

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