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Two Views of the Projective Plane

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SENIOR THESIS APPROVAL

This Honors thesis entitled

"Two Views of the Projective Plane"

written by

Rebecca J. Thomas

and submitted in partial fulfillment of the
requirements for completion of the
Carl Goodson Honors Program
meets the criteria for acceptance
and has been approved by the undersigned readers.

Steve Hennagin, thesis director

Jeffrey Sykes, second reader

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April 11, 2005

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Two Views of the Projective Plane

Rebecca J. Thomas

April 15, 2005

0.1 Abstract

The projective plane is a mathematical object which can be defined in two ways. In the following paper, I will explain the two definitions and show how they are equivalent by establishing a homeomorphism between the two objects.

0.2 Introduction and Explanation of Key Terms

0.2.1 Equivalence Relations

One of the definitions of the projective plane uses an equivalence relation. An equivalence relation is a relation that is reflexive, symmetric, and transitive.

The simplest example of an equivalence relation is the equals sign, "=." The equals sign is a relation because it demonstrates a relationship between two objects. For example, $4 = 4$ or $1 + 3 = 4$.

The equals sign is an equivalence relation because the relation is reflexive, i.e. an object "equals" itself: $4 = 4$. The equals sign is symmetric because the order of the two objects being compared can be reversed: $1 + 3 = 4$ automatically means $4 = 1 + 3$. Finally, the equals sign is transitive because "equals" can be transferred. For example, if you know $1 + 3 = 4$ and $4 = 2 + 2$, then you also know that $1 + 3 = 2 + 2$.

0.2.2 Geometry and Topology

The other definition of the projective plane is topological. In this paper, I will be concerned with the topological properties of the two objects I am comparing. Topological properties are the properties of an object that are preserved under deformation. If I can establish a homeomorphism between the two projective planes, the topological properties will be preserved, and the two objects can be called equivalent.

Geometric properties are those which are not preserved by deformation. For instance, one definition of the projective plane causes it to have spherical geometry. The geometry of the other definition is another matter entirely.

0.2.3 *Mobiüs* Bands

I will mention these curious objects briefly in my discussion on orientation. "*Mobiüs* band" is another name for the familiar *Mobiüs* strip.

0.3 The Projective Plane as an Equivalence Relation

The projective plane is a partition of $\mathbf{R}^3$minus the origin into all lines passing through but not including the origin. What follows is a formal definition:

Let $T = \{(x,y,z) \in \mathbf{R}^3 \mid (x,y,z) \neq (0,0,0)\}$. Define a relation, $\sim$, on T by $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ if there is a nonzero real number λ such that $x_1 = \lambda x_2$, $y_1 = \lambda y_2$, and $z_1 = \lambda z_2$.

The concept of this projective plane, which will be referred to as $T/\sim$ is well-defined, as the following proposition shows:

Proposition 0.3.1 *$\sim$ is an equivalence relation on T.*

To show $a \sim a$, where $a \in \mathbf{R}^3$, let $\lambda = 1$.

Suppose $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ by λ. Let $\delta = \lambda^{-1}$. Then, $x_2 = \delta x_1$, $y_2 = \delta y_1$, and $z_2 = \delta z_1$. So, $(x_2, y_2, z_2) \sim (x_1, y_1, z_1)$.

Suppose $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ by λ_1 and $(x_2, y_2, z_2) \sim (x_3, y_3, z_3)$ by λ_2. Then, $x_1 = \lambda_1 x_2$ and $x_2 = \lambda_2 x_3$. Then, $x_1 = \lambda_1(\lambda_2 x_3)$. Let $\theta = \lambda_1\lambda_2$, a nonzero real number. Then, $x_1 = \theta x_3$. Similarly, $y_1 = \theta y_3$, and $z_1 = \theta z_3$. Therefore, $\sim$ is transitive. Thus, it is an equivalence relation. □

0.3.1 A Note on the Geometry of the Equivalence Classes

The equivalence class $[x, y, z]$ is a line in $\mathbf{R}^3$with the origin removed that passes through (x, y, z) and $(0, 0, 0)$. Consider the vector $< x, y, z >$ with initial point $(0, 0, 0)$ and ending point (x, y, z). Vectors with a common initial point are collinear if they are parallel, that is, if they are equal up to a scalar multiple. Then, any two $a, b \in \mathbf{R}^3$are collinear if they are in the same equivalence class (the scalar multiple is λ), and all points on one of these lines, excepting the origin, are in the equivalence classes.

Thus, the equivalence relation partitions the space into an infinite number of lines radiating from the origin. These equivalence classes are called the points of the projective plane. See Figure 1 for an illustration of a few of the equivalence classes, points, in $T/\sim$.

0.4 The Projective Plane as a Surface

The projective plane is a hemisphere with antipodal (opposite) points on the rim glued. It is a closed, homogenous, and non-orientable surface with local spherical geometry. It will be referred to as $\mathbf{P}^2$. In Figure 2, the gluing is represented by a dotted line on one side of the rim and a solid line on the other side.

0.4.1 A Note on Gluing and Orientation

Gluing is the abstract connection of points. It is important to note that physical connection of the points need not be possible, and indeed this is the case with the projective plane. In Figure 3, two sets of antipodal points, points at either end of a diameter, are shown with lines connecting the points that will be connected by the gluing. Note that the two lines cross, a demonstration of a property that makes the projective plane so interesting.

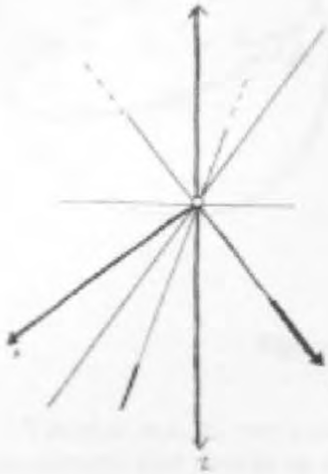

Figure 1: The Projective Plane, i.e. $T/\sim$.

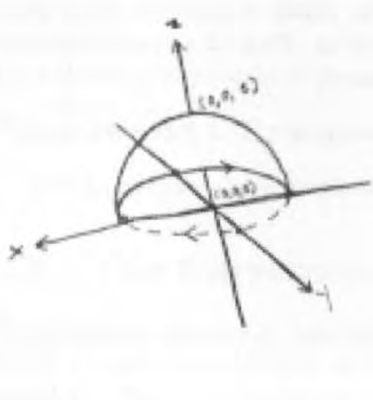

Figure 2: The Projective Plane, i.e. $\mathbf{P}^2$.

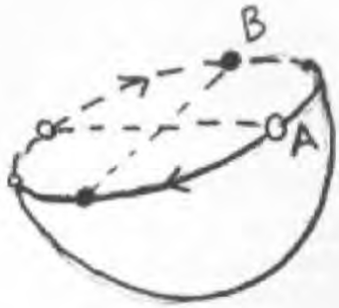

Figure 3: Two Antipodal Points

Visualize that the two points labelled "A" and "B" are the endpoints of the line segment that runs along the rim from A to B counterclockwise. (Please see the discussion on lines in a later section.) If you look at the line segment from the side, point B is on the right, and point A is on the left. Imagine moving the line segment towards the antipodal points. (Of course, this operation is instantaneous.) The two endpoints switch places. Now, slide the line segment around the spherical surface and back to its original location. Point A is now on the right, and point B is on the left. (This is demonstrated in the picture by the different colors of the endpoints.) This curious result of the gluing is represented in Figure 3 and Figure 2 by a set of arrows, one on either side of the rim, pointing in opposite directions.

A path that an object can slide along and return with the left- and right-hand sides reversed is called orientation-reversing. If a surface contains an orientation-reversing path, a $Mobiüs$ band, it is called non-orientable. Thus, the following proposition is proven:

Proposition 0.4.1 *The projective plane is non-orientable.*

The projective plane contains a $Mobiüs$ band. □

0.5 The Correspondence

The following steps will form the basis for the correspondence: From $T/\sim$, select a representative from each equivalence class. Let this set of points be called S'. For each $(x, y, z) \in S'$ choose instead the point $(\lambda x, \lambda y, \lambda z)$, where $\lambda^{-1} = \sqrt{x^2 + y^2 + z^2}$. Note that all points now in S' lie on the unit sphere in $\mathbf{R}^3$centered at the origin because of the value of λ. Form the set S from S' as follows:

- If for (x, y, z), $z > 0$, let (x, y, z) be the representative point in S.

- If for (x, y, z), $z = 0$ and
  1. $y > 0$, let $(x, y, 0)$be the representative point.
  2. $y < 0$, pick $(-x, -y, 0)$.
  3. $y = 0$, let $(x, 0, 0)$ be the point.
- If for (x, y, z), $z < 0$, pick $(-x, -y, -z)$.

Note that in $\mathbf{R}^3$, the set S forms a hemisphere in the upper four octants with the boundary from and including the point on the $-x$-axis through the $-y$-axis up to the positive x-axis (but not including that point) removed. Since nothing except multiplying components by real, nonzero values has been done to the members of S', S also contains a single representative from each equivalence class.

Note that S matches the description of $\mathbf{P}^2$. The boundary points included in S are equivalent under $\sim$ to the antipodal points. In this sense, the rim points are glued.

Proposition 0.5.1 *$\mathbf{P}^2$ is homeomorphic to $T/\sim$.*

Let $f : \mathbf{P}^2 \to T/\sim$ such that $f((x, y, z) \in \mathbf{P}^2) = [x, y, z] \in T/\sim$. Suppose $f((a, b, c)) = f((x, y, z))$. Then, $[a, b, c] = [x, y, z]$. But, S contains only one point from each equivalence class, so λ relating (a, b, c) to (x, y, z) in $T/\sim$ must be 1. Therefore, f is one-to-one.

Let $[x, y, z]$ be a member of $T/\sim$. I have shown that it is possible to find (x_1, y_1, z_1) related to (x, y, z) by $\sim$ that is in S. Then, $f((x_1, y_1, z_1)) = [x_1, y_1, z_1] = [x, y, z]$. f is therefore onto.

So, f is a one-to-one correspondence between $\mathbf{P}^2$ and $T/\sim$.

In order for f to be a homeomorphism, it must be continuous, and it's inverse must be continuous. In order for the function to be continuous it must send open sets to open sets, and the proof of this presents a problem because we have not yet defined what the open sets of $T/\sim$ are.

Consider an open ball in $\mathbf{P}^2$. By the mapping, the open ball maps to an open cone in $T/\sim$, which is a logical definition of an open set. Please see Figure 4 for an example of an open ball in $T/\sim$. Defining the open cones as open balls, both f and its inverse, which exists because f is a one-to-one correspondence, are continuous.

So, f is a homeomorphism between $\mathbf{P}^2$ and $T/\sim$. □

0.6 Lines

A great circle is defined as a circle which is the intersection of a sphere with a plane through its center. In this case, the *sphere* is $\mathbf{P}^2$ and the planes are those through the origin.

In normal three-dimensional space, $ax + by + cz = 0$ is the general equation of a plane through $(0, 0, 0)$ with direction $< a, b, c >$. By selecting appropriate

Figure 4: A Solid Open Cone Is Defined to be an Open Ball

representatives (the ones comprising S), one can see that the intersection of one of these planes with $\mathbf{P}^2$is a great circle on $\mathbf{P}^2$.

For great circles to be lines, they must be the shortest distance between two points in $\mathbf{P}^2$.

Proposition 0.6.1 *Great circles are the lines in* $\mathbf{P}^2$.

On a normal sphere in regular space, a great circle contains the antipodal points (x, y, z) and $(-x, -y, -z)$ because $xa + yb + ac = 0$ implies that $-xa - yb - zc = 0$. So, opposite points on a great circle always lie on a diameter. In $\mathbf{P}^2$, the opposite points of the diameter are identified, and so the great circle is a loop.

Let x and y be two points in $\mathbf{P}^2$. A great circle between them has curvature $k = 1/r$, where r is the distance from the center of the hemisphere, the origin.

Any point on the great circle defined by x and y has the property that $r = 1$, so k=1. To prove that a great circle is a line, it is sufficient to prove that is has minimum curvature.

Note that if we defined an alternate "line" which traverses the "interior" of $\mathbf{P}^2$, then r is smaller. Therefore, k would be larger. This is not a line.

Therefore, the great circles are the lines. This implies that $\mathbf{P}^2$has spherical geometry. □

Proposition 0.6.2 $\mathbf{P}^2$*has local spherical geometry.*

Great circles are lines in $\mathbf{P}^2$. □

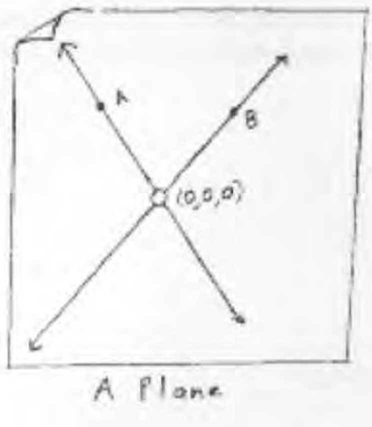

Figure 5: A Line in $T/\sim$, Shown in Normal Space as a Plane

In order to show that lines in $\mathbf{P}^2$map to lines in $T/\sim$, I must prove that these planes minus the origin in $\mathbf{R}^3$can be defined as lines in the projective plane, $T/\sim$. I will show that they are well-defined. Geometrically, if we choose different points to be the representatives of the two equivalence classes, will we still get the same plane in $\mathbf{R}^3$?

Proposition 0.6.3 *$L = \{[x, y, z] \in T/\sim |ax + by + cz = 0\}$ is well-defined.*

Let (a, b, c) be in T. Suppose (x_1, y_1, z_1), (x_2, y_2, z_2) are in T and $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$. If $ax_1 + by_1 + cz_1 = 0$, then $a\lambda x_1 + b\lambda y_1 + c\lambda z_1 = 0$, then $ax_2 + by_2 + cz_2 = 0$. So, these two points in the equivalence set define the same plane in $\mathbf{R}^3$. □

See Figure 5. Note that A, B, and the origin define the plane. We have just proven that any A or B in their equivalence classes may be selected, and the same plane will be defined.

Proposition 0.6.4 *Two distinct lines in $T/\sim$ always intersect at one point in $T/\sim$.*

Given two distinct lines $ax + by + cz = 0$ and $px + qy + rz = 0$ in $T/\sim$, their intersection point is $[br - cq, cp - ar, aq - bp]$. This is easy to check. □

So, the plane definition is a valid definition for the lines in $T/\sim$. I have mentioned that lines in $\mathbf{P}^2$are the intersection of planes in $\mathbf{R}^3$containing the origin with the hemisphere that we mapped points to in the correspondence. These planes in $\mathbf{R}^3$are lines in $T/\sim$. Thus, as expected lines map to lines.

0.7 Conclusion

In this paper, I have defined and explained the two definitions of the projective plane and explored some interesting properties of each. I have proven that the two definitions of the projective plane, $T/\sim$ and $\mathbf{P}^2$, are equivalent and shown that the correspondence maps lines to lines.

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